The complex determined by a congruence and a line

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Abstract

A construction is presented to find all lines of a linear complex given by a linear congruence and a line. In addition a method is given to get an image of such a complex 'moving along the congruence'.

References

[Stoss] Hanns-Jörg Stoss, *Einführung in die synthetische Liniengeometrie*, Dornach 1999

[Ziegler] Renatus Ziegler, *Synthetische Liniengeometrie*, Dornach 1981

Conventions and notations

We will use the following theorem and symbols.

Theorem Let $\mathcal{G}$ be a linear congruence and l a line not in it. $\mathcal{G}$ and l determine exactly one linear complex $\mathcal{C}(\mathcal{G}, l)$ that contains them.
For a proof see [Stoss] p. 83ff.

symbol	meaning	example
$\wedge$	meet	$l \wedge \alpha$ is the meeting point of line l and plane α
$\vee$	join	$P \vee l$ is the plane that joins P and l
$\langle P, \alpha \rangle$	the pencil of lines in α through P	

1 Construction

Let $\mathcal{G}$ be a linear congruence and l a line not in it and let $\mathcal{C} = \mathcal{C}(\mathcal{G}, l)$ be the linear complex determined by them. We want to find all the lines of $\mathcal{C}$. We know that each point of space contains exactly one pencil of lines of $\mathcal{C}$, and so does each plane.

To start with the simplest case, suppose $\mathcal{G}$ is elliptic. Then through every point of space passes exactly one line of $\mathcal{G}$. This is true also for the points of l. The lines of $\mathcal{G}$ that meet l compose a regulus $\mathcal{R}_l$. For each $m \in \mathcal{R}_l$ the pencil $\langle l \wedge m, l \vee m \rangle$ is part of $\mathcal{C}$. So for each plane containing l we found the corresponding pencil of the complex, and also for each point on l we found its pencil.

Now let A be any point of space not on l. Then there is exactly one line g of $\mathcal{G}$ containing A.

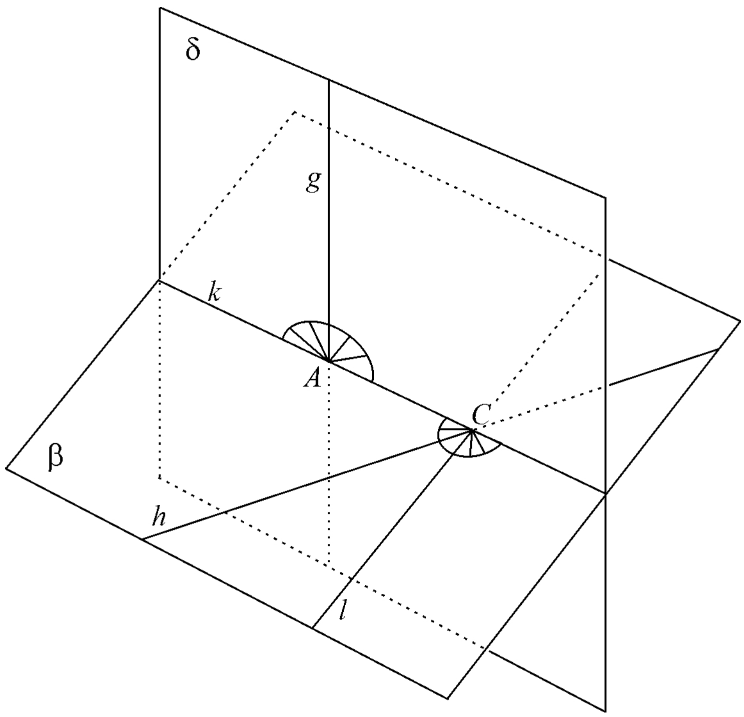

Figure 1: The pencil in an anarbitrary point A

- Suppose g and l are skew, see figure 1. Let $\beta = A \vee l$. This plane contains exactly one line h of the congruence, and this line meets l in a point C. The pencil $\langle C, \beta \rangle$ belongs to our complex, and one of its lines is $CA = k$. Now we have two lines g and k in A that belong to the complex. Let $\delta = g \vee k$, then the pencil of complex lines in A is $\langle A, \delta \rangle$.
- If g meets l in a point E, then take any point $F \neq E$ on l and let g' be the line of $\mathcal{R}_l$ through F. Take a third line l' from the pencil $\langle F, l \vee g' \rangle$, i.e. $l' \neq l$ and $l' \neq g'$. Now apply the previous construction with l' instead of l.

Dually, let α be any plane not containing l. Then there is exactly one line g of $\mathcal{G}$ in α.

- Suppose g and l are skew, see figure 2. Let $B = \alpha \wedge l$. Through this point passes exactly one line $h =$ of the congruence, and joining this line with l gives a plane $\gamma = h \vee l$. The pencil $\langle B, \gamma \rangle$ belongs to our complex, and one of its lines is $\gamma \wedge \alpha = k$. Now we have two lines g and k in α that belong to the complex. Let $D = g \wedge k$, then the pencil of complex lines in α is $\langle D, \alpha \rangle$.
- If g and l are incident, we can apply the dual of the above trick and find a line l' not in $\mathcal{G}$. Next apply the previous construction with l' instead of l.

The reader will have noticed that the two figures are essentially the same. If in our second construction we had started with α in the position of plane δ of the first, the two figures would have coincided.

If $\mathcal{G}$ is parabolic with directrix d, and if l and d are skew, then we have the same construction as above, as long as $A \not< d \not< \alpha$.

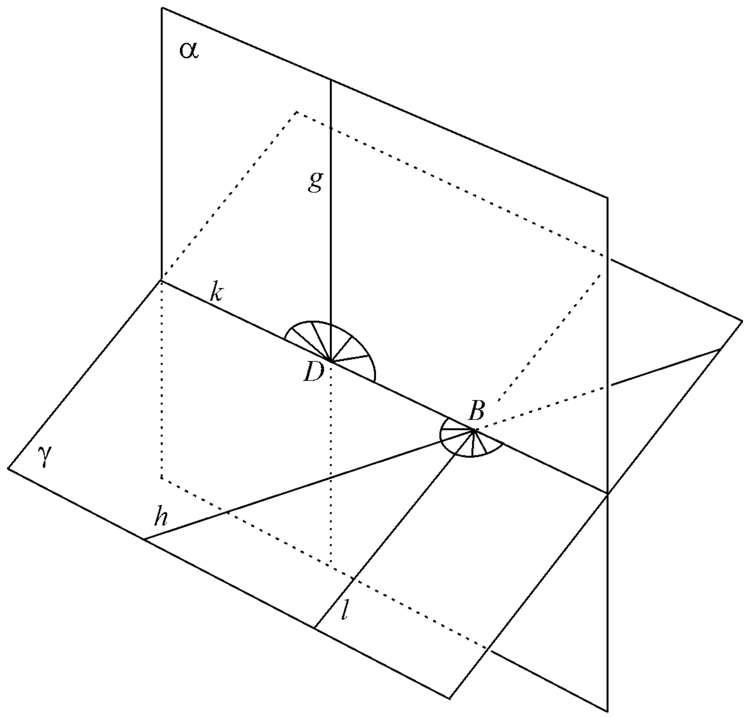

Figure 2: The pencil in an anarbitrary plane α

If $\mathcal{G}$ is hyperbolic with directrices d and d', and if l is skew to both directrices, then too we have this same construction, provided again that A and α are on neither directrix.

If $\mathcal{G}$ is special, i.e. a line field in γ and a line bundle through C, and if l meets γ in P, then $\mathcal{C}$ is special with directrix PC.

The reader is invited to investigate the remaining configurations.

2 A pencil of complexes

In addition to the previous, we can get an image of the *pencil* of complexes through $\mathcal{G}$. To achieve this, we will put our figures into motion: we'll move our complex along $\mathcal{G}$ by moving our initial line l to a new position l'. Now, we can move a line in two different ways: either turn it about a point in a plane, or move it to a skew position. In both cases it is possible to do the constructions of the previous section with the new line l'. However, if we remember that through every point of l passes one line of $\mathcal{G}$ - they form the regulus $\mathcal{R}_l$ - and that the lines of this congruence remain in their position, turning l about a point in a plane would not be very realistic. So we'll move l to l' along the regulus $\mathcal{R}_l$. All lines x of $\mathcal{C}$ that do not belong to $\mathcal{G}$ will then move along their regulus $\mathcal{R}_x$.

So, take two more points on l and draw the lines of $\mathcal{G}$ through them, see figure 3. One of them is m. Take any point B' on h and construct the unique transversal l' of $\mathcal{R}_l$ through B'. In moving l up to l', α will turn about g to $\alpha' = g \vee B'$ and β will turn about h to $\beta' = h \vee l'$. In our figure P is the meeting point of m and α. With that the meeting point A' of β' and g is constructed, and that is the point where A moves to. To find the image n' of an arbitrary line $n \in \langle A, \alpha \rangle$ we need one more line of $\mathcal{G}$, viz. of $\mathcal{R}_n$. This line meets α' in a point Q' and

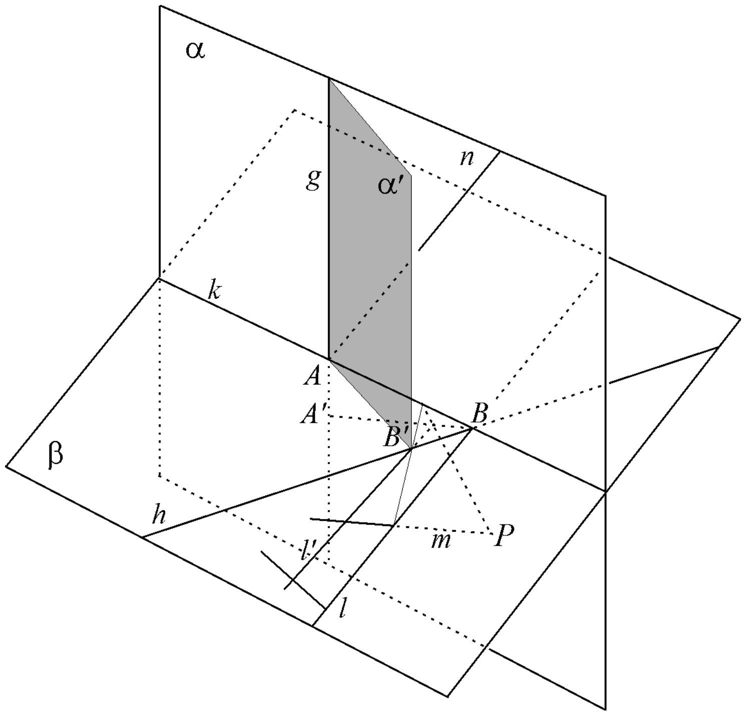

Figure 3: Moving complex

now $n' = A'Q'$. The last step is not done in our figure, but the reader is recommended to complete the drawing.

As a last exercise one could try to perform this construction in a 'real' elliptic congruence like figure 4. The long thin regulus is supposed to be inside the short broad one, and 'concentric'. The thick lines are supposed to be in front of the thin ones. Either can be seen as transversals of the others, by the way. Draw in this figure one extra line, e.g. l, that is supposed to be in the front side of the outer regulus. Construct the pencil in an arbitrary point or plane, e.g. A, that is supposed to be in the front side of the inner one. Veryfy that $l \vee h$ passes approximately through A, using rotational symmetry. Then move line l along the outer regulus and follow how the pencil $\langle A, \alpha \rangle$ moves and how a line of this pencil moves.
The meaning of this exercise is not so much drawing exactly points and lines, as well as imagining lines and pencils to move simultaneously. Good luck!

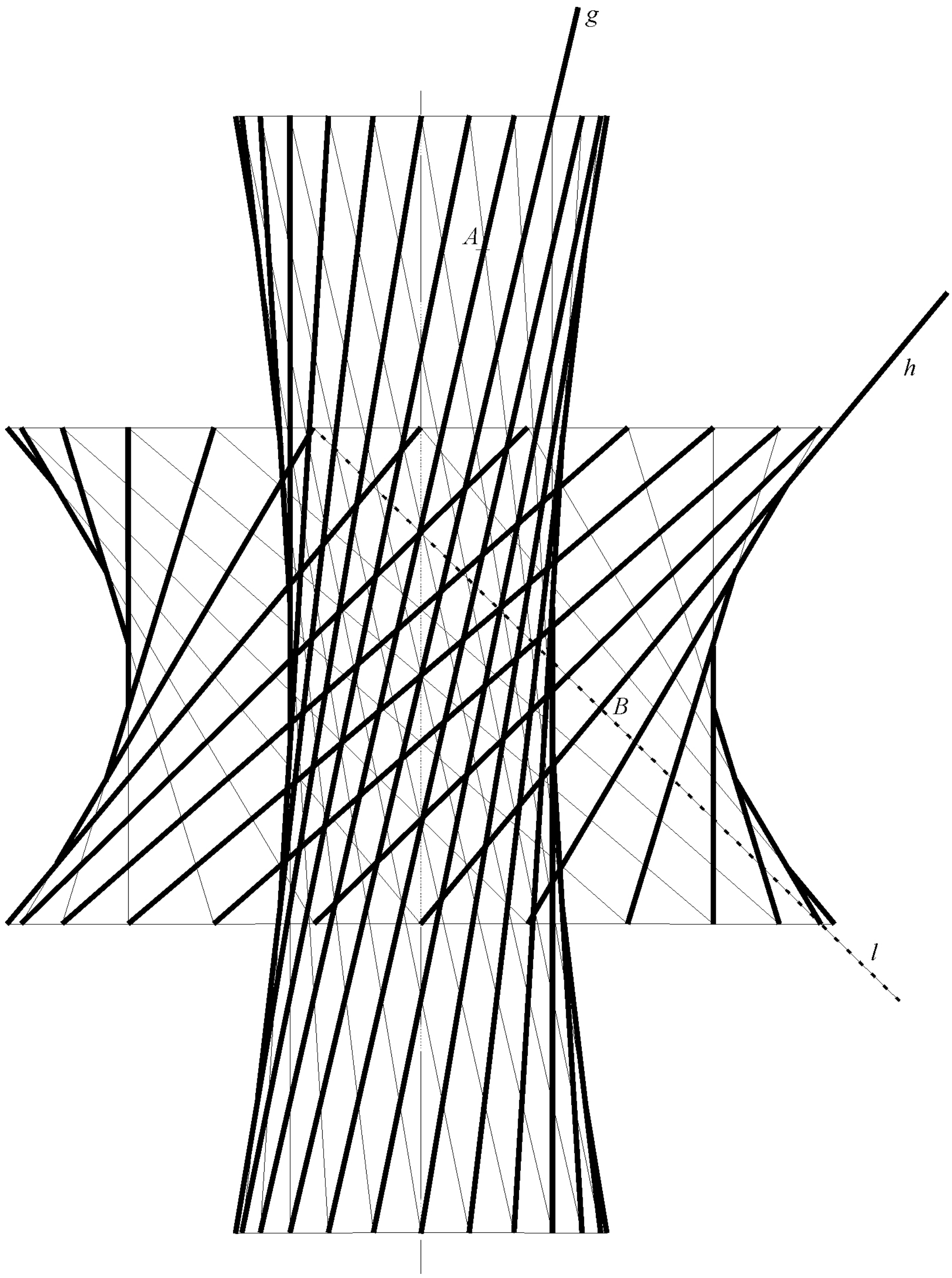

Figure 4: Exercise