Classification of real projective Pathcurves

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Preface

Pathcurves were first investigated by Felix Klein and Sophus Lie in 1871 (see [6]). The topic did not meet much interest until George Adams in the 30s and Lawrence Edwards after World War II studied the occurrence of these curves in forms of living nature. The work of Edwards (see [3]) triggered me to dig into the matter. I reformulated the work of Klein and Lie with Linear Algebra, and tried to extend it to the three-dimensional case.

The reader is supposed to be familiar with elementary Projective Geometry as well as with Linear Algebra and Matrix Theory. Concerning the latter, in for instance [7] the reader will find almost all he needs for this article, in particular a treatise on canonical matrix-forms. For the basics of Projective Geometry one can consult for instance [1].

Specially the investigation of the 3-dimensional curves has been tough, and the work would not have been possible without the support, comments and corrections of Gerard Hermans, Ruud Pellikaan and Pepe Veugelers, to whom I like to express my gratitude.

Introduction

Consider the n-dimensional projective space $\mathbf{P}_n$ over the *real* numbers. Its *elements* are points (of dimension zero), lines (1-dimenional), planes (2-dimensional)... but also the empty set $\emptyset$ (dimension -1) and the whole space itself. The elements are partially ordered: $\mathbf{a} \leq \mathbf{b}, \mathbf{b} \geq \mathbf{a}$, $\mathbf{a}$ is contained in $\mathbf{b}$, $\mathbf{b}$ contains $\mathbf{a}$, all mean the same. As usual we accept $\mathbf{a} \leq \mathbf{a}$. $\emptyset$ is the minimal and $\mathbf{P}_n$ the maximal element. The *join* $\mathbf{a} \vee \mathbf{b}$ of two elements is the smallest element containing both $\mathbf{a}$ and $\mathbf{b}$. Their *meet* $\mathbf{a} \wedge \mathbf{b}$ is the biggest element that is contained in both. So the join of a point and a line is - in general - a plane, and their meet is empty. With these conventions a projective space is a *lattice*. A map $P : \mathbf{P}_n \longrightarrow \mathbf{P}_n$ is said to be projective if meet and join are preserved, i.e. if

$$P(\mathbf{a} \vee \mathbf{b}) = P(\mathbf{a}) \vee P(\mathbf{b}) \quad \text{and} \quad P(\mathbf{a} \wedge \mathbf{b}) = P(\mathbf{a}) \wedge P(\mathbf{b})$$

for any two elements $\mathbf{a}, \mathbf{b}$. A projective map P is necessarily bijective. A map is projective if and only if it is bijective and it preserves order:

$$\mathbf{a} \leq \mathbf{b} \Longleftrightarrow P(\mathbf{a}) \leq P(\mathbf{b})$$

A projective map is fully determined by the images of a basis. Given a fixed coordinate system in $\mathbf{P}_n$, every projective map is characterized by any of a family of regular $(n+1) \times (n+1)$-matrices $k \cdot A$, where k is any non-zero real number. We will identify P and these matrices, so we will write $P = A = kA$.

Repeated application of P to some starting point $\mathbf{x}_0$ (not being an eigenvector of P) produces a series of different points

$$\mathbf{x}_1 = P(\mathbf{x}_0), \quad \mathbf{x}_2 = P^2(\mathbf{x}_0), \quad \ldots$$

Taking the inverse P^{-1} of P, we get the points

$$\mathbf{x}_{-1} = P^{-1}(\mathbf{x}_0), \quad \mathbf{x}_{-2} = P^{-2}(\mathbf{x}_0), \quad \ldots$$

The points $\mathbf{x}_i$ are on a so called *pathcurve*[1]. The map $Q = P^2$, applied to $\mathbf{x}_0$ repeatedly, defines the same pathcurve as P, but the motion has double 'speed'. In many cases[2] one can *define* $R = P^v$ for any real number v. If $v \neq 0$, again, repeated application of R defines the same pathcurve as P, but with 'velocity' v relative to P. In these cases $P^{vt}(\mathbf{x}_0)$ can be considered as the locus at time t of a point that moves with 'speed' v along the pathcurve defined by P and $\mathbf{x}_0$; here t can be any real number.

0.1 Non-integer powers of a matrix

Let P be a regular $n \times n$ matrix over the reals. From linear algebra (i.c. Jordan decomposition) we know that there exist matrices M, N, Q and S and an integer $m \leq n$ such that $Q =$

[1] The German Felix Klein called it a *W-Kurve*, which was misunderstood as *Weg-Kurve*, and then translated as *pathcurve*.

[2] If one allows imaginary numbers these powers are well-defined for every non-singular map.

$M^{-1}PM = S(I+N)$, where $S = (\alpha_i \delta_{ij})$ is a diagonal-matrix, containing the (possibly complex) eigenvalues of P, and $N^m = 0$. I is the identity.

The matrix N vanishes if all eigenvalues have multiplicity 1. If in $\mathbf{P}_2$ a is an eigenvalue of say multiplicity 3, then either

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or $N = 0$ (the remaining possibility is equivalent with the second). For real t define $S^t = (\alpha_i^t \delta_{ij})$ and

$$(I+N)^t = \sum_{i=0}^{m-1} \begin{pmatrix} t \\ i \end{pmatrix} N^i$$

the right part being the binomial development (which has a finite number of terms because $N^m = 0$). Finally define $P^t = MQ^tM^{-1} = MS^t(I+N)^tM^{-1}$. It is immediately clear that with these definitions the following properties of powers also hold for matrices:

$$P^aP^b = P^{a+b} \quad \text{and} \quad (P^a)^b = P^{ab}$$

0.2 Classification criteria

A *pathcurve* is a parameter curve $\mathbf{v} = P^t\mathbf{a}, \mathbf{a}$ not being an eigenvector of P (so the identity is excluded). Obviously this curve is invariant under each power of P. Two curves $P^t\mathbf{a}$ and $P^t\mathbf{b}$ either coincide or are disjoint. The *pathcurve system* of this map is the collection of all curves that are invariant under P^t for each real t.

Clearly, the eigenspaces are also invariant under each power of P, so classification starts with studying eigenspaces. Let the map P have characteristic polynomial $\Pi_k {p_k}^{a_k}$. The factors are arranged such that the degree of p_k is not smaller than that of p_l whenever $k > l$, and if they have equal degree, then $a_k \geq b_k$. Also we assume p_k irreducible over the reals. So the ordering in the product is from high degree to low degree, and within the same degree from high exponent to low exponent. A second map Q has characteristic polynomial $\Pi_k {q_k}^{b_k}$, with similar ordering.

Now P and Q are said to have the same *eigenspace structure* if for each k

- $a_k = b_k$, and
- the degrees of p_k and q_k are equal.

To put it short: apart from the coefficients within the factors, the polynomials are equal. Having the same eigenspace structure is an equivalence relation.

Two maps P and Q are said to have the same *shape* if there exists a regular projective map A and a real $v \neq 0$ such that $Q = A^{-1}P^vA$. P^v and Q are called *similar*. Clearly A maps every pathcurve $\mathbf{v} = Q^t\mathbf{a}$ on a pathcurve $\mathbf{w} = P^{vt}A\mathbf{a}$. A *shape* is an equivalence class of this relation. Similar maps have equal characteristic polynomials; maps with equal shapes have the same eigenspace structure.

As a consequence, if P and Q have different eigenspace structures, they represent different shapes. That the converse is not true will be shown in the sequel. So 'shape' is a refinement of 'eigenspace structure'.

Two systems P and Q are said to have the same *position* it there is a real $v \neq 0$ such that $Q = P^v$. In this case v is called the *speed* of Q with respect to P; if $v > 0$, the maps are said to have *equal senses*, else they have *opposite senses*. Evidently, the pathcurves of P and Q (or rather their orbits) coincide.

Classification by 'shape' is very detailed. In § 2.3.2 for instance, we encounter a one-parameter family of shapes that are spirals for all but one value of the parameter. For this exeptional value, however, a set of 'concentric conics' appears. All these shapes have the same eigenspace structure of one real and two imaginary eigenvectors. They differ in the 'curvature'.
So, at the other hand, classification by eigenspace structure is not detailed enough. In this example we want a classification that takes into account the topological property of being closed.
In § 3.3.2 we'll find a *projective* difference between two sets of shapes, again with equal eigenspace structures.
Hence, there is a need for a third criterion, which groups together different shapes - and splits up eigenspace classes - in geometric types. Intuitively this seems to be the most interresting classification. However, until now we did not succeed in finding an algebraic definition of 'type'. But may be we should not want to, and accept that geometry is *not* only a part of algebra.

Summarizing we have the following classification tree of projective maps:

eigenspace structure - geometric type - shape - position - sense

0.3 Eigenspaces

So, our main concern is to investigate the invariant elements of the defining maps, i.e. finding eigenvalues and eigenspaces of the corresponding matrices. Invariant points are the eigenvectors of the pathcurve map P, i.e. the solutions for $\mathbf{v}$ of $(P - uI)\mathbf{v} = 0$, where u is any eigenvalue and I the identity matrix.

Sometimes we may want to look at a transpose $M^{-1}PM$ rather than P itself. We have $M^{-1}PM - uI = M^{-1}PM - M^{-1}uIM = M^{-1}(P - uI)M$ so $(M^{-1}PM - uI)\mathbf{v} = \mathbf{0}$ has the same solutions as $M^{-1}(P - uI)M\mathbf{v} = \mathbf{0}$. But the last one is equivalent with $(P - uI)M\mathbf{v} = \mathbf{0}$ because M is regular. So $\mathbf{v}$ is a solution of the first equation, if and only if $M\mathbf{v}$ is a solution of the last one.

Let be given an arbitrary $(n-1)$-dimensional subspace $\mathbf{m}$ and let $\mathbf{n} = (P^\tau)^{-1}\mathbf{m}$. Then $P^\tau\mathbf{n} = \mathbf{m}$ and $\mathbf{n}^\tau P = \mathbf{m}^\tau$. Now take any point $\mathbf{a}$ of $\mathbf{m}$, so $\mathbf{m}^\tau\mathbf{a} = 0$, and let $\mathbf{b} = P(\mathbf{a})$. Then $\mathbf{n}^\tau\mathbf{b} = \mathbf{n}^\tau P(\mathbf{a}) = \mathbf{m}^\tau\mathbf{a} = 0$, so every point of $\mathbf{m}$ maps into a point of $\mathbf{n}$: the inverse of the transpose of P is the extension of P to the set of $(n-1)$-dimensional subspaces. The *invariant* subspaces are the solutions for $\mathbf{v}$ of

$$((P^{\tau})^{-1} - \frac{1}{u}I)\mathbf{v} = 0$$

(To get the images of for instance lines in space, more sophisticated tools are available, see [4] and [5].)

0.4 Intervals and duality

If $\mathbf{a} \leq \mathbf{b}$ the interval $[\mathbf{a}, \mathbf{b}]$ consists of all elements between $\mathbf{a}$ and $\mathbf{b}$:

$$[\mathbf{a}, \mathbf{b}] = \{\mathbf{x} | \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}$$

Its length is $\dim \mathbf{a} - \dim \mathbf{b}$; its dimension is one less than its length. So if A is a point in some plane α, $[A, \alpha]$ is the pencil of lines through A in α. It has length 2 and dimension 1.

A duality on an interval $[\mathbf{a}, \mathbf{b}]$ of dimension ≥ 2 is a bijective map that reverses order:

$$\mathbf{a} \leq \mathbf{b} \Longleftrightarrow P\mathbf{a} \geq P\mathbf{b}$$

A duality interchanges join and meet. The dual of $[\mathbf{a}, \mathbf{b}]$ is denoted by $[\mathbf{a}, \mathbf{b}]^*$. Now in real projective space we find the following non-trivial geometries:

1 dimensional: $[\emptyset, l]\,, [A, \alpha]\,, [l, \mathbf{P}_3]$

2 dimensional: $[\emptyset, \alpha]\,, [\emptyset, \alpha]^*\,, [A, \mathbf{P}_3]\,, [A, \mathbf{P}_3]^*$

3 dimensional: $[\emptyset, \mathbf{P}_3]\,, [\emptyset, \mathbf{P}_3]^*$

where α is a plane containing line l, which in turn contains point A. In the sequel, motions of points (in a line, plane or space) are studied, but the results are easily translated into the other geometries of equal dimension.

0.5 Conventions

Points are given in brackets, like $(x_0 : \cdots : x_n)$, and will be identified with column vectors

$$\mathbf{x} = \begin{pmatrix} kx_0 \\ \vdots \\ kx_n \end{pmatrix}, \quad k \neq 0$$

The $(n-1)$-dimensional subspaces (lines in $\mathbf{P}_2$, planes in $\mathbf{P}_3$) are given in square brackets, $[x_0 : \cdots : x_n]$.

The *orbit* of a moving point $\mathbf{v}_t = \mathbf{f}(t)$ is the set $\{\mathbf{f}(t) | t \in \mathbf{R}\}$.
If $\lim_{t\to-\infty} \mathbf{f}(t) = \mathbf{v}_{-\infty}$ exists, then $\mathbf{v}_{-\infty}$ is called the *source* of the curve. We also say that the curve *originates* in $\mathbf{v}_{-\infty}$. If $\lim_{t\to\infty} \mathbf{f}(t) = \mathbf{v}_{\infty}$ exists, then $\mathbf{v}_{\infty}$ is called the *sink* of the curve. We also say that the curve *terminates* in $\mathbf{v}_{\infty}$. If all or 'almost all' pathcurves have the

same source or sink, this point is also called the source/sink of the system or of the motion. Clearly, source and sink of a pathcurve are eigenvectors of the defining map.

If the components $f_i(t)$ of $\mathbf{v}_t$ are differentiable at $t = t_0$ we denote $\dot{\mathbf{v}}(t_0) = \mathbf{f}'(t_0) = (f_i'(t_0))$, which is a point of the tangent to the curve at $\mathbf{v}(t_0)$. Similarly $\ddot{\mathbf{v}}$ is the second derivative, etc. In 3-dimensional space the equation of the osculating plane is

$$\det(\mathbf{x},\ \mathbf{v},\ \dot{\mathbf{v}},\ \ddot{\mathbf{v}}) = 0$$

Reals are printed in roman type. Greek letters denote imaginary numbers.

0.6 Numbers

In the projective plane there are seven continuous types of pathcurves, of which two contain only straight lines.

In 3-dimensional projective space we found 27 different types of continuous pathcurve-systems. Of these, five are made up of straight (semi-) lines. Seven types contain true plane curves but no twisted ones. The remaining 15 types are true space systems[3].

	points	straight lines	plane curves	twisted curves
$\mathbf{P}_1$	1	3	0	0
$\mathbf{P}_2$	1	2	5	0
$\mathbf{P}_3$	1	5	7	15

Below we present the results of the classification in deatil. Section 1 gives some preliminary considerations on projective maps of the line. A major part is dedicated to finding the powers of a matrix with two imaginary eigenvalues. Section 2 deals with plane pathcurves, and of course with invariant points and lines. In these two sections we also consider maps that cannot be made continuous. In space (section 3) however, we restrict to continuous maps. Also some invariant surfaces will be considered. After the *summary* you'll find the figures.

[3]Baldus [2] found 14 types of collineations of the *complex* projective space. He did not study pathcurves in detail.

1 The projective line

This section gives an overview of the regular linear maps of the real projective line. If our map is represented by the matrix

$$P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

its characteristic equation is

$$(a - x)(d - x) - bc = x^2 - (a + d)x + (ad - bc) = 0$$

Depending on the value of the discriminant

$$\Delta = (a + d)^2 - 4(ad - bc) = (a - d)^2 + 4bc$$

there are 1 or 2 (complex) solutions, i.e. eigenvalues.

We have the following cases.

- $(x - p)^2 = 0,\ p \neq 0$. The matrix is similar to either

  $$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \text{ or } \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}$$

  see § 1.1 and § 1.2.

- $(x - p)(x - q) = 0,\ p \neq q,\ pq \neq 0$. The matrix is similar to

  $$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

  see § 1.3.

- $(x - \lambda)(x - \bar{\lambda}) = 0$, see § 1.4.

If there are two different real invariant points, they split up the projective line in 2 *intervals*.

> We will use the Riemann sphere as a model for the complex line (the 1-point compactification of the Gaussian plane). From complex calculus we mention that linear transformations of the extended Gaussian plane map circles (including straight lines, being circles with infinite radius) onto circles.

If our map has two different (real or imaginary) eigenvectors $\mathbf{a}$ and $\mathbf{b}$ and if $Q = P^v$, it appears that the speed v is the quotient of logarithms of *cross ratios*:

$$v = \frac{\ln(\mathbf{abx}Q\mathbf{x})}{\ln(\mathbf{abx}P\mathbf{x})}$$

for every non-eigenvector $\mathbf{x}$.

1.1 Identity

The matrix is similar to

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$P = P^t$ is the identity. Every point is invariant, so there are no pathcurves.

1.2 Parabolic map

The matrix is similar to

$$P = \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$$

where $v = p^{-1} \neq 0$, so

$$P^t = \begin{pmatrix} 1 & vt \\ 0 & 1 \end{pmatrix}$$

v is the speed of the system with respect to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

If $v > 0$, the motion is from (1:0) via (-1:1), (0:1), (1:1) towards (1:0). If $v < 0$, it is the other way round. Source and sink coincide in invariant point (1:0). All maps have the same shape. This case is the intermediate between those of the next two sections.

> In the extended Gaussian plane points move on lines parallel to the real axis under our P. In the general case they move on circles that touch the real axis in its invariant point.

1.3 Hyperbolic map

The matrix can be written in the following form.

$$P = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

If $|p| > |q|$, the source is (0:1), the sink (1:0); else the motion has opposite direction.

- If one parameter is negative P^t is only defined for integer values of t. The points are jumping from one interval to the other, or, more precisely: for a general point $\mathbf{x}$ the points $P^n\mathbf{x}$ are in one interval if n is even, and in the other if n is odd.

> Extending to the complex line also this 'jumping' case can be made continuous. The points move on loxodromes (spirals) from source to sink. [4]

[4] Rudolf Steiner (1861-1925), without knowing this, at on hand associated the imaginary with the astral, and at the other hand mentioned that what is a linear movement in the real world, becomes a spiral one in the astral realm. A most remakable consistency!

- If $q = -p$ then the map is an involution. Though there are still two invariant points, there is neither source nor sink.

  In the Gaussian plane points now move on concentric circles around O; or, in the general case, on Apollonian circles around the invariant points.

- If both p and q are positive the motion remains within one interval. We have

  $$P^t = \begin{pmatrix} p^t & 0 \\ 0 & q^t \end{pmatrix}$$

  Without loss of generality we can take $\det P = pq = 1$. So alternatively we can put

  $$P^t = \begin{pmatrix} e^{vt} & 0 \\ 0 & e^{-vt} \end{pmatrix}$$

  where $v = \ln p \neq 0$ is the speed. There is only one shape.

  In the Gaussian plane the pathcurves are half lines from O to ∞. Or in general, they build a pencil of circles through the two invariant points; a single pathcurve is a segment of such a circle between source and sink.

- If both parameters are negative we multiply with -1 to get the previous case.

1.4 Elliptic map

The discriminant

$$\Delta = (a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc$$

of the characteristic equation

$$x^2 - (a+d)x + ad - bc = 0$$

of our map

$$P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is negative. This implies $\det P = ad - bc > 0$ and $bc < 0$. The roots are:

$$\lambda = \frac{a+d}{2} + i\frac{\sqrt{-\Delta}}{2} = re^{iv}, \qquad \bar{\lambda} = \frac{a+d}{2} - i\frac{\sqrt{-\Delta}}{2} = re^{-iv}$$

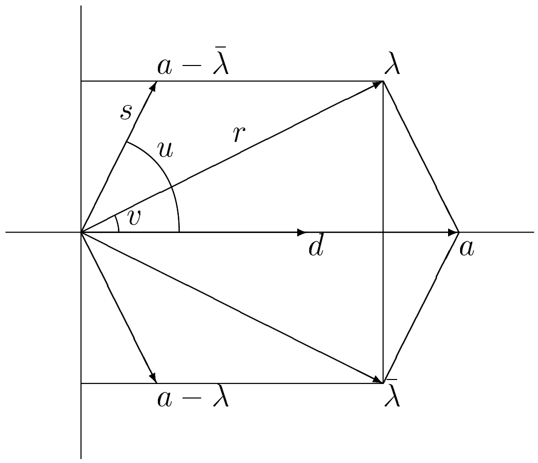
where

$$r = \sqrt{ad-bc} = \sqrt{\det P}$$

$$v = \begin{cases} \arctan \frac{\sqrt{-\Delta}}{a+d} & \text{if } a+d > 0 \\ \pi/2 & \text{if } a+d = 0 \\ \pi + \arctan \frac{\sqrt{-\Delta}}{a+d} & \text{if } a+d < 0 \end{cases}$$

Note that $v \in \langle 0, \pi \rangle$, so $v > 0$.

Clearly there are no real invariant points.



1.4.1 Continuity

Independent eigenvectors of P are for instance

$$\begin{pmatrix} c \\ \lambda - a \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} c \\ \bar\lambda - a \end{pmatrix}$$

or

$$\begin{pmatrix} a - d \pm i\sqrt{-\Delta} \\ 2b \end{pmatrix}$$

Putting $a - \bar\lambda = se^{ui}$ (see figure, it appears that $s = \sqrt{-bc}$) we transpose with

$$M = \begin{pmatrix} c & c \\ \lambda - a & \bar\lambda - a \end{pmatrix} = \begin{pmatrix} c & c \\ -se^{-ui} & -se^{ui} \end{pmatrix}$$

Now we have

$$M^{-1}PM = \begin{pmatrix} \lambda & 0 \\ 0 & \bar\lambda \end{pmatrix}$$

so for any real number t

$$P^t = M \begin{pmatrix} \lambda^t & 0 \\ 0 & \bar\lambda^t \end{pmatrix} M^{-1}$$

This real matrix can be written as

$$P^t = \frac{r^t}{\sin u} \begin{pmatrix} \sin(u+vt) & \frac{c}{\sqrt{-bc}} \sin vt \\ \frac{b}{\sqrt{-bc}} \sin vt & \sin(u - vt) \end{pmatrix}$$

where

$$u = \arg(a - \bar\lambda) = \arctan \frac{\sqrt{-\Delta}}{a-d} \quad \text{or} \quad u = \pi + \arctan \frac{\sqrt{-\Delta}}{a-d}, \qquad u \in \langle 0, \pi \rangle$$

Taking ‘small’ motions (i.e. $t \approx 0$, but positive) it turns out to be that

- if $b > 0$, then the motion is one way round ('to the left', in the direction $(1:0) \to (1{:}1) \to (0:1)$);
- if $b < 0$, then the motion is the other way round ('to the right').

In both cases the line is traversed infinitely many times for $t \in \mathbf{R}$.

P is an involution iff $a + d = 0$.

1.4.2 Speed and shape

The matrix

$$M = \begin{pmatrix} 1 & \frac{a-d}{2b} \\ 0 & 1 \end{pmatrix}$$

moves $(0:1)$ to the real part of the eigenvectors, $((a-d)/2b : 1)$. Now transpose our map by M:

$$M^{-1}PM = \begin{pmatrix} \frac{a+d}{2} & \frac{\Delta}{4b} \\ b & \frac{a+d}{2} \end{pmatrix}$$

Clearly, without loss of generality we can take $a = d$. Also, if necessary we can multiply our matrix with -1, so we will assume $a \geq 0$. As a consequence $v \in \langle 0, \pi/2]$. So we have

$$P = \begin{pmatrix} a & c \\ b & a \end{pmatrix}$$

Also $\Delta = 4bc < 0$ and $u = \pi/2$. The eigenvectors $(\pm qi : 1)$ are purely imaginary.

Next, transpose our map by:

$$M_1 = \begin{pmatrix} 1 & -1 \\ q & q \end{pmatrix}$$

where $q = \sqrt{-b/c}$. Then

$$M_1^{-1}PM_1 = \begin{pmatrix} a & -b/q \\ b/q & a \end{pmatrix}$$

So we can also take $b + c = 0$ without loss of generality:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Finally, we can divide by $r = \sqrt{\det P} = \sqrt{a^2 + b^2}$. Now our map becomes

$$P = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \quad \text{if } b > 0, \text{and}$$

$$P = \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix} \quad \text{if } b < 0$$

If we allow v to assume negative values then we have

$$P = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \quad P^t = \begin{pmatrix} \cos vt & -\sin vt \\ \sin vt & \cos vt \end{pmatrix}$$

where v is defined as follows:

$$v = \begin{cases} \frac{b}{|b|} \arctan \frac{|b|}{a} & \text{if } a > 0 \\ \pi/2 & \text{if } a = 0 \end{cases} \qquad v \in \langle -\pi/2, \pi/2] \setminus \{0\}$$

remembering that $a \geq 0$.

Clearly all elliptic maps have the same shape. The eigenvalues are now $\cos v \pm i \sin v$ and the eigenvectors are $(\mp i : 1)$

> Though there are two invariant points in the complex line there is neither source nor sink. The points move in Apollonian circles around the invariant points. This is easily seen by computing the path of $(pi : 1)$. This point moves on the circle with centre $(p^2+1)/2p$ and radius $(p^2-1)/2p$.

In the sequel we will frequently have to transpose our last real matrix P to its complex normalized one. Let

$$M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

i.e. an eigenvector matrix. Then

$$M^{-1}PM = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos v + i \sin v & 0 \\ 0 & \cos v - i \sin v \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

i.e. an eigenvalue matrix.

2 Plane pathcurves

Plane pathcurves are the curves that are invariant under non-trivial linear maps P of the projective plane. The characteristic equation of such a map is of degree 3 and has at least one real root. Apart from the identity we have the following cases.

- $(x-p)^3 = 0,\ p \neq 0$. The matrix is similar to either of the following ($v = p^{-1} \neq 0$):

$$\begin{pmatrix} p & 0 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$$

  see § 2.1.1; *or*:

$$\begin{pmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{pmatrix} = \begin{pmatrix} 1 & v & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$$

  see § 2.1.2.

- $(x-p)^2(x-q) = 0,\ p \neq q,\ pq \neq 0$. The matrix is similar to either of the following: ($a = qp^{-1} \neq 0, 1; v = p^{-1} \neq 0$):

$$\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

  see § 2.2.1; *or*:

$$\begin{pmatrix} p & 1 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \end{pmatrix} = \begin{pmatrix} 1 & v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

  see § 2.2.2.

- $(x-a)(x-b)(x-c) = 0,\ a, b, c$ all different and non-zero. The matrix is similar to

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

  see § 2.3.1.

- $(x-\lambda)(x-\bar{\lambda})(x-r) = 0,\ r \neq 0$. From the previous section we know that the matrix is similar to

$$\begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & r \end{pmatrix}$$

  see § 2.3.2.

In this section lines are identified with row vectors in square brackets like $[a : b : c]$.

We use the following named points: the origin $O(0 : 0 : 1)$, $X(1 : 0 : 0), Y(0 : 1 : 0)$. The x-axis is the line $[0 : 1 : 0]$ through O and X, the y-axis $[1 : 0 : 0]$ through O and Y; $\infty = XY = [0 : 0 : 1]$ is the 'line at infinity'.

If there are n invariant non-concurrent lines, they divide the plane in 2^{n-1} disjoint *regions* ($n = 1, 2, 3$). Each continuous non-straight pathcurve is completely within one region.

2.1 One eigenvalue

2.1.1 Elation

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \quad P^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & vt \\ 0 & 0 & 1 \end{pmatrix}, \qquad v \neq 0$$

The point $Y(0 : 1 : 0)$ is linewise invariant, the line $\infty[0 : 0 : 1]$ is pointwise invariant. The pathcurves are straight lines through $Y(0 : 1 : 0)$ carrying parabolic motions; Y is source as well as sink of the entire system. There is one shape only; v is the speed. See figure **??**. In the affine case P is a translation parallel to the y-axis. It concerns type V of the Klein/Lie-classification [6].

2.1.2 Contangential conics

$$P = \begin{pmatrix} 1 & v & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \quad P^t = \begin{pmatrix} 1 & vt & \frac{1}{2}t(t-1)v^2 \\ 0 & 1 & vt \\ 0 & 0 & 1 \end{pmatrix}, \qquad v \neq 0$$

The line $\infty[0 : 0 : 1]$ and the point $X(1 : 0 : 0)$ are invariant. On this line there is a parabolic motion. The remaining pathcurves are conics (parabolas in the affine case) that touch the line at infinity in X, which is source and sink of the system. There is one region and one shape only; v is the speed. See figure **??**. It concerns type III of the Klein/Lie-classification.

Changing v seems to change the shape. However, let

$$A = \begin{pmatrix} 1 & \frac{1}{2}w & \frac{1}{8}vw \\ 0 & 1 & \frac{1}{2}v \\ 0 & 0 & 1 \end{pmatrix}$$

and $u = vt/w$. Then

$$Q^u = A^{-1}P^tA = \begin{pmatrix} 1 & wu & \frac{1}{2}u(u-1)w^2 \\ 0 & 1 & wu \\ 0 & 0 & 1 \end{pmatrix}$$

so P and Q have the same shape indeed.

2.2 Two different eigenvalues

2.2.1 Homology

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \neq 0, 1$$

The line $\infty[0:0:1]$ is pointwise invariant. The point $O(0:0:1)$ is linewise invariant. If $|a| > 1$ O is sink of the system, else it is its source.

- Suppose $a < 0$ but $a \neq -1$. A general point $\mathbf{x}$ and its image $\mathbf{x}' = P\mathbf{x}$ are on one line l with O. If the common point of l and ∞ is called $\mathbf{x}_0$, then $\mathbf{x}$ and $\mathbf{x}'$ separate O and $\mathbf{x}_0$. So repeated application of P leads to points jumping between the two intervals of l determined by $\mathbf{x}_0$ and O.

  In the complex plane this point moves on a loxodrome from source to sink within the complex line l.

- If $a = -1$ we have our map is an involution (reflection in ∞) and $\mathbf{x}, \mathbf{x}'$ are harmonic with respect to $\mathbf{x}_0, O$.

  In the complex plane points now move on Apollonian circles within the complex lines.

- If $a > 0$ (but $a \neq 1$), the motion can be made continuous.

  $$P^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{vt} \end{pmatrix}$$

  where $v = \ln a \neq 0$. The pathcurves are straight semi-lines between O and ∞. There is only one shape; v is the speed. See figure **??**.
  It concerns type IV of the Klein/Lie-classification.

2.2.2 Logarithms

$$P = \begin{pmatrix} 1 & v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \neq 0, 1; \; v \neq 0$$

These systems are degenerate instances of the ones in §2.3.1. They arise when either source or sink coincides with the third point of the invariant triangle. It concerns type II of the Klein/Lie-classification. There are two invariant points, $X(1:0:0)$ and $O(0:0:1)$. Also there are two invariant lines: $\infty[0:0:1]$ and $x[0:1:0]$. There are two regions.

- If $a > 0$ (but $a \neq 1$) then

$$P^t = \begin{pmatrix} 1 & vt & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{svt} \end{pmatrix}$$

where $s = \frac{\ln a}{v}$. The line at infinity carries a parabolic motion, the x-axis a hyperbolic one. After a change of basis, the affine equation of a pathcurve can be put in the form

$$y = p + q \ln x$$

which explains the name. Also *exponential pathcurve* is used. Each curve is in one of two regions.
Here for the first time we find different shapes, determined by the parameter s (v is the speed).

If s and v are positive then the source is X, the sink O. Increasing s makes the curve stay longer near the parabolic line and to avoid the hyperbolic one. See figure **??**.

- If $a < 0$, $a \neq -1$, a general point jumps from one region to the other: in each region they are on a pathcurve of the previous case, which is called a *branch* of the discrete curve.

  In the complex plane points now move on loxodrome-like curves from source to sink.

- If $a = -1$, these branches are the straight lines of the elation (see § 2.1.1) with centre X and axis OX which carries an involution.

  In the complex plane a general point moves along a spiral on a cone with apex X.

2.3 Three different eigenvalues

This case is type I of the Klein/Lie-classification.

2.3.1 All real: triangular system

- Assume first a, b and c positive (and different).

$$P = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad P^t = \begin{pmatrix} a^t & 0 & 0 \\ 0 & b^t & 0 \\ 0 & 0 & c^t \end{pmatrix}, \quad abc \neq 0, \text{all different}$$

This is the general form of the real plane pathcurve. Each vertex and line of the fundamental triangle is invariant, which is why we call it the *triangular* system. Each curve is within one of four regions. The motion is from the eigenvector with the smallest eigenvalue towards the one with the biggest. If one of the eigenvalues is the geometric mean of the other two (e.g. $b^2 = ac$), then the curves are 'half' conics. Without loss of generality we can assume $a > b = 1 > c$. If we take $v = \ln a$ as the speed, $s = -\frac{\ln c}{\ln a} > 0$ determines the shape:

$$P^t = \begin{pmatrix} e^{vt} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-svt} \end{pmatrix}$$

See figure **??**. If $s = 1$ we have the above mentioned conics. If $s > 1$ the curves are 'skew' to one side: they are blunter near X and sharper near O. If $s < 1$ they are skew to the other side. For $s \to \infty$ and for $s \to 0$ the curves degenerate into the straight lines of the homology (see § 2.2.1).

- If one of the parameters is negative, for instance $c < 0, c \neq -a$, then t must be integer. The points jump from one region to the other: the points within one region are again on a pathcurve of the previous type, a *branch* of the discrete curve.

  In the complex projective plane the points are moving on loxodrome-like curves from source to sink.

- If e.g. $a > b = 1, c = -a$ the branches are the straight lines of the homology (§ 2.2.1).

  In the complex projective plane a general point moves along a spiral on a half cone.

- The remaining cases can be reduced to the previous ones by multiplying with -1.

2.3.2 One real, two imaginary

From §1.4.2 we conclude that the matrix is similar to

$$P = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & r \end{pmatrix} = M \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \bar{\lambda} & 0 \\ 0 & 0 & r \end{pmatrix} M^{-1}, \quad v \in \langle -\pi/2, \pi/2 \rangle \setminus \{0\}$$

where

$$M = \begin{pmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\lambda = \cos v + i \sin v$. If $r < 0$ we multiply by -1; then by adding $\pm\pi$ to v this minus sign is removed from the trigonometric functions. So we can assume r to be positive if we allow v to have values in $\langle -\pi, \pi \rangle$ exept 0. Putting $s = (\ln r)/v$ we have

$$P = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & e^{sv} \end{pmatrix}, \quad P^t = \begin{pmatrix} \cos vt & -\sin vt & 0 \\ \sin vt & \cos vt & 0 \\ 0 & 0 & e^{svt} \end{pmatrix} v \in \langle -\pi, \pi \rangle \setminus \{0\}$$

There is one invariant real point, $O(0:0:1)$, and one invariant real line $\infty[0:0:1]$; hence only one region.

As usual, v is the speed of the system, s is the shape. One could use 'angular' speed for v as opposed to the 'radial' speed $sv = \ln r$. The shape determines the 'curvature' (big s resulting in almost straight lines).

There are two different types.

- If $s = 0$ the invariant curves are 'concentric' conics (see figure **??**), which are traversed infinitely many times.

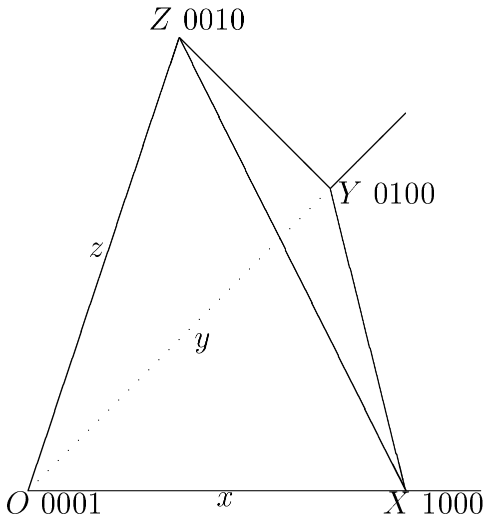
- Else they are (logarithmic) spirals (see figure **??**). If s is positive, the curves are *inward* spirals with common sink O; for $t \to -\infty$ they stretch more and more and approach the line at infinity. For negative shapes O is common source of *outward* spirals.

The limiting case $s = \pm\infty$, or equivalently $v = 0$, is the homology of § 2.2.1.

3 Pathcurves in space

We use the coordiates $(x_0 : \cdots : x_3)$ for points, and $[u_0 : \cdots : u_3]$ for planes. Since Plücker-coordinates for lines are not generally known we will not use them. However, in determining (limiting) tangents they play an important role.

The subspace $x_3 \neq 0$ we will call the affine space, so $x_3 = 0$ represents the plane at infinity, denoted by ∞.



We use the following named points: the origin $O(0:0:0:1)$, $X(1:0:0:0), Y(0:1:0:0), Z(0:0:1:0)$. The x-axis is the line through O and X, the y-axis through O and Y, and the z-axis through O and Z. The unit point (1:1:1:1) is inside the above tetrahedron.

3.0.3 The characteristic function

The characteristic function of a real 4x4-matrix P is a polynomial of degree 4. Imaginary roots appear in pairs of conjugate complex numbers, so the polynomial cannot be irreducible: it factors in at least two quadratic ones with real coefficients. Since kP represents the same projective map as P for every real non-zero k, we can take one of the real eigenvalues equal to 1, without loss of generality.

We distinguish the following cases.

1 eigenvalue

* $(x-1)^4 = 0$; see 3.1

2 eigenvalues

* $(x-a)(x-1)^3 = 0$; see 3.2.1 - 3.2.3
* $(x-a)^2(x-1)^2 = 0$; see 3.2.4 - 3.2.6

* $(x^2+px+q)^2 = (x-\lambda)^2(x-\bar{\lambda})^2 = 0$; see 3.2.7 - 3.2.9

3 eigenvalues

* $(x-a)(x-b)(x-1)^2 = 0$; see 3.3.1, 3.3.2
* $(x^2+px+q)(x-1)^2 = (x-\lambda)(x-\bar{\lambda})(x-1)^2 = 0$; see 3.3.3, 3.3.4

4 eigenvalues

* all real: $(x-1)(x-a)(x-b)(x-c) = 0$; see 3.4.1
* two real, two imaginary, see 3.4.2:

$$(x^2+px+q)(x-a)(x-b) = (x-\lambda)(x-\bar{\lambda})(x-a)(x-b) = 0$$

* all imaginary, see 3.4.3:

$$(x^2+px+q)(x^2+rx+s) = (x-\lambda)(x-\bar{\lambda})(x-\mu)(x-\bar{\mu}) = 0$$

If a pathcurve is entirely in one plane, it is called a *plane* curve, otherwise a *twisted* one.

If there are n invariant planes, they split the space in 2^{n-1} disjoint *regions* $(n = 1, 2, 3, 4)$. Each twisted curve remains within a single region.

3.0.4 Names

Several curves (e.g. the helix) are well known. Others are less known, or at least were not known to me until now. Of these, some ask for obvious names, like *cone spiral*. Five curves have been baptized by me; with the help of English, German and French colleagues I also provided names in their languages. For brevity we often omitted the adjectives *real, projective* and *twisted*.

3.0.5 Figures

We also tried to visualize the twisted curves. In general we plotted one single representative curve from some system; often we added the motions on (some of) the invariant lines. In a few cases we presented a series of curves of one system, in order to visualize a surface. Big circles represent ‘nearby’ points, small circles remote ones.

3.1 One eigenvalue

3.1.1 Elation

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & vt \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad v \neq 0$$

The plane $\infty[0:0:0:1]$ is pointwise invariant (and - of course - linewise too), the point Z(0:0:1:0) is planewise invariant, i.e. every plane through Z is mapped onto itself. The pathcurves are straight lines through Z each carrying a parabolic motion. Z is source and sink of the entire motion. There are no twisted curves. There is one shape only, and v is the speed of the system.
In the affine case P represents a translation parallel to the z-axis.

3.1.2 Parabolic congruence

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^t = \begin{pmatrix} 1 & vt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & vt \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad v \neq 0$$

The line XZ is point- and planewise invariant.
The plane through X, Z and $(0:p:0:q)$ contains a set of straight pathcurves, each carrying a parabolic motion and converging to $(p:0:q:0)$, like in figure **??**. We have a *linear congruence*, namely a parabolic one. There are no twisted curves. The invariant lines have 2 degrees of freedom (there are ∞^2 invariant lines). There is one shape only, and v is the speed.

3.1.3 Plane contangential conics

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & vt & \frac{1}{2}v^2t(t-1) \\ 0 & 0 & 1 & vt \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad v \neq 0$$

The line XY is pointwise invariant, the line YZ planewise.
Exept for the plane ∞ [0:0:0:1], every plane through YZ contains a set of conics (parabolas in the affine case) that all touch YZ in Y (like in figure **??**). YZ carries a parabolic motion. When such a plane turns towards infinity, the conics become 'sharper'. The limiting plane ∞[0:0:0:1] contains a pencil of straight lines through Y. Y is source and sink of the system.

There are no twisted curves. There is only one shape, v is the speed.

3.1.4 Cubics

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & v & 0 \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^t = \begin{pmatrix} 1 & vt & \frac{1}{2}v^2t(t-1) & \frac{1}{6}v^3t(t-1)(t-2) \\ 0 & 1 & vt & \frac{1}{2}v^2t(t-1) \\ 0 & 0 & 1 & vt \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad v \neq 0$$

There is one invariant point $X(1:0:0:0)$, one invariant plane, ∞ $[0:0:0:1]$ and one invariant line XY.
The whole affine space is filled with twisted cubics each of which originates and terminates in X (see figure **??**). The cubics all have XY as tangent and ∞ as osculating plane. In turn, ∞ contains a set of conics all touching XY in X (like in figure **??**). XY cariies a parabolic motion.
There is only one shape; v is the speed of the system.

Changing v seems to change the shape. However, let

$$A = \begin{pmatrix} 1 & \frac{1}{3}(2w-v) & 0 & 0 \\ 0 & 1 & \frac{1}{6}(v+w) & 0 \\ 0 & 0 & 1 & \frac{1}{3}(2v-w) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $u = vt/w$. Then $Q^u = A^{-1}P^tA =$

$$\begin{pmatrix} 1 & wu & \frac{1}{2}w^2u(u-1) & \frac{1}{6}w^3u(u-1)(u-2) \\ 0 & 1 & wu & \frac{1}{2}w^2u(u-1) \\ 0 & 0 & 1 & wu \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, again, P and Q have the same shape. Compare the remark at the end of § 2.1.2.

3.2 Two different eigenvalues

3.2.1 Homology

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad a \neq 0, 1$$

The point O(0:0:0:1) is planewise invariant, the plane ∞ $[0:0:0:1]$ is pointwise invariant. In the affine case the map P is a central multiplication from O with factor a.

If $a > 0$ the map can be made continuous and we have

$$P^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{vt} \end{pmatrix}$$

where $v = \ln a$. The pathcurves are straight semi-lines through O, carrying hyperbolic motions. There is one shape only, and v determines the speed of the motion. If $v < 0$ (i.e. $0 < a < 1$) the motion originates at O and terminates in ∞; otherwise otherwise. There are no twisted curves.

3.2.2 Plane logarithms, I

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \ a \neq 0, 1; \ v \neq 0$$

Eigenvalue 1 leads to a pointwise invariant line XZ and a planewise invariant line OX. Eigenvalue a gives O as eigenvector and ∞ as invariant plane. The lines in ∞ through X are invariant, and so are the lines in OXZ through O.

If $a > 0$ the map can be made continuous.

$$P^t = \begin{pmatrix} 1 & vt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{svt} \end{pmatrix}$$

where $s = \frac{\ln a}{v}$. Every plane through OX contains a set of logarithmic pathcurves like the ones in figure **??**, the second line (with the parabolic motion) being in ∞. When such a plane turns towards OXZ, the curves stretch from O outward. OXZ in turn contains a pencil of straight lines through O with a pointwise invariant line XZ, like the homology of figure **??**. The plane ∞ carries an elation with centre X and axis XZ (figure **??**). The shape of the system is determined by s, v is the speed. There are no twisted curves.

3.2.3 Conic turns

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \ a \neq 0, 1; \ v \neq 0$$

There are two invariant points, $O(0:0:0:1)$ and $X(1:0:0:0)$, two invariant planes $OXY[0:0:1:0]$ and ∞, and two invariant lines OX and XY.

If $a > 0$ the map can be made continuous.

$$P^t = \begin{pmatrix} 1 & vt & \frac{1}{2}v^2t(t-1) & 0 \\ 0 & 1 & vt & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{svt} \end{pmatrix}$$

where $s = \frac{\ln a}{v}$. There is a hyperbolic motion on OX, a parabolic one on XY. In ∞ we have a set of conics all touching XY in X, like in figure **??**. In OXY we have a set of logarithmic pathcurves like in figure **??**. The rest of the projective space is filled with beautiful twisted curves like in figure **??**. Each curve turns *once* around a semi-cone, which is why we call them *conic turn* (Ge: *Kegelschlag*; Fr: *tour de cône*; Du: *kegelslag*). Each cone has apex O and

one of the conics of ∞ as its base; so the cones all touch OXY in OX.
If $a > 1$ and $v > 0$, X is the source, O the sink of the entire motion. The shape is determined by s, v is the speed of the system[5]. The invariant lines and planes are (limiting) tangents resp. osculating planes of each twisted curve. Each curve is within one of two regions. Figure **??** shows a set of conic turns on one semi-cone.

3.2.4 Hyperbolic congruence

$$P = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; a \neq 0, 1$$

The lines OZ and XY are pointwise and planewise invariant.

If $a > 0$ the map can be made continuous.

$$P^t = \begin{pmatrix} e^{vt} & 0 & 0 & 0 \\ 0 & e^{vt} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $v = \ln a$. All pathcurves are straight semi-lines joining OZ and XY. There is one shape only, and v is the speed.
Again we encounter a linear congruence, this time a hyperbolic one. There are no twisted curves.

3.2.5 Plane logarithms, II

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \; a \neq 0, 1; \; v \neq 0$$

The line OZ is pointwise invariant, the line XY planewise. Furthermore X and OXZ are invariant. OXZ contains a pencil of invariant straight lines through X (homology).

If $a > 0$ the map can be made continuous.

$$P^t = \begin{pmatrix} 1 & vt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{svt} & 0 \\ 0 & 0 & 0 & e^{svt} \end{pmatrix}$$

where $s = \frac{\ln a}{v}$. Each plane through XY contains a set of logarithmic pathcurves like in figure **??**. Note the difference with §3.2.2: there the invariant planes have the line with the two double points in common; here it is the line with one double point. The shape is determined by s, v is the speed. There are no twisted curves.

[5] See remark at the end of § 2.1.2.

3.2.6 Long twisted logarithms

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & v \\ 0 & 0 & 0 & a \end{pmatrix}, \; a \neq 0, 1; \; v \neq 0$$

X, Z, OZ, XY, XZ, OXZ and ∞ are invariant.

If $a > 0$ the map can be made continuous.

$$P^t = \begin{pmatrix} 1 & vt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a^t & vta^{t-1} \\ 0 & 0 & 0 & a^t \end{pmatrix} = \begin{pmatrix} 1 & vt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{svt} & vte^{sv(t-1)} \\ 0 & 0 & 0 & e^{svt} \end{pmatrix}$$

where $s = \frac{\ln a}{v}$. The planes OXZ and ∞ each contain a set of logarithmic pathcurves like in figure **??**. All other pathcurves are twisted curves, like in figure **??**, each within one of only *two* regions. So they are twice as 'long' as the ones in §3.3.2 (each curve meets four of the eight regions in which the fundamental tetrahedron divides space). The shape is determined by s, v is the speed. Suppose $a < 1$ and $v > 0$. Then the motion originates in Z in the direction OZ, turns through space in one long bow, and terminates in X, direction XY. Hence OZ and XY are tangent to each curve, and these are skew. The planes OXZ and ∞ are (limiting) osculating planes of each curve.

3.2.7 Elliptic congruence

$$P = M \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} M^{-1} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$$

where

$$Q = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}, \quad v \in \langle -\pi/2, \pi/2 \rangle \setminus \{0\}$$

(see § 1.4).

$$P^t = M \begin{pmatrix} \lambda^t & 0 & 0 & 0 \\ 0 & \overline{\lambda}^t & 0 & 0 \\ 0 & 0 & \lambda^t & 0 \\ 0 & 0 & 0 & \overline{\lambda}^t \end{pmatrix} M^{-1} = \begin{pmatrix} Q^t & 0 \\ 0 & Q^t \end{pmatrix}$$

where

$$Q^t = \begin{pmatrix} \cos vt & -\sin vt \\ \sin vt & \cos vt \end{pmatrix}$$

There are no real invariant points, nor planes. The image of an arbitrary point $A(a_0 : a_1 : a_2 : a_3)$ is

$$A(t) = \cos vt \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} + \sin vt \begin{pmatrix} -a_1 \\ a_0 \\ -a_3 \\ a_2 \end{pmatrix}$$

so the pathcurves are straight lines with elliptic motions, all with the same speed v.
The pathcurves constitute a third type linear congruence, the elliptic one.
There are no twisted curves. There is only one shape.

3.2.8 Impossible case

Since the matrix

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix}$$

is not invariant under complex conjugation this case cannot occur in real projective space.

3.2.9 Sheafs

$$P = M \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & \bar{\lambda} & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \bar{\lambda} \end{pmatrix} M^{-1} = \begin{pmatrix} Q & R \\ 0 & Q \end{pmatrix}$$

Where

$$Q = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}, R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \quad r > 0$$

(see 1.4.2). The last matrix commutes with every other $2 \times 2-$matrix. There are no real invariant points, nor planes. There is one invariant line: XY.

$$P^t = \begin{pmatrix} Q^t & R(t) \\ 0 & Q^t \end{pmatrix}$$

where

$$Q^t = \begin{pmatrix} \cos vt & -\sin vt \\ \sin vt & \cos vt \end{pmatrix}$$

and

$$R(t) = rt \begin{pmatrix} \cos v(t-1)) & -\sin v(t-1) \\ \sin v(t-1) & \cos v(t-1) \end{pmatrix}$$

The shape is $s = r/v$.

It is not so easy to get an image of this system. Therefore we describe one curve in more detail (see figure **??**). Let $v = 0.01, r = 1$ and let our curve be defined by

$$\mathbf{v} = P^t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

If $t \approx -\infty$ the orbit is an almost straight line very close but slightly skew to XY. If t increases by one period π we are on the next part of the orbit, which is sligthly more skew and more remote from XY. So, in the neighbourhood of $t = -\infty$ we get a set of (almost straight) lines on a hyperboloid-like surface.

Now let t further increase. Distance from XY and skewness increase, as does 'curvature'. Suddenly, in the neighbourhood of $t = 0$, the orbit becomes an almost planar semi-circle near the plane OYZ. Then gradually it stretches, coming more and more in the direction of XY again, but now approaching from the other side.

The complete picture - of one sigle orbit - looks like a sheaf of corn tied together by the semi-circle, which is why we call it the *sheaf* (Ge: *Garbe*; Fr: *gerbe*; Du: *schoof*). See figure ??.

3.3 Three different eigenvalues

3.3.1 Plane triangular systems

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad a, b \notin \{0, 1\}, a \neq b$$

All the points, lines and planes of the fundamental tetrahedron are invariant. In addition, XY is pointwise and OZ is planewise invariant. Finally the lines through O in OXY and the lines through Z in ∞ are invariant.

If a and b are positive the map can be made continuous.

$$P^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{vt} & 0 \\ 0 & 0 & 0 & e^{svt} \end{pmatrix}$$

where $v = \ln a$ and $s = \frac{\ln b}{v}$. Each plane through OZ contains a triangular sytem of pathcurves like in figure **??**. In ∞ as well as in OXY the pathcurves are like in figure **??**. The shape is determined by s, v is the speed. One can distinguish two cases:

- $s < 0$ (1 is between a and b): the motion is between O and Z.
- Else ($s > 0$) the motion is between XY and either O or Z.

There are no twisted curves.

3.3.2 Short twisted logarithms

$$P = \begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad a, b \notin \{0, 1\}, a \neq b, v \neq 0$$

O, X, Z, OXZ, OXY and ∞ are invariant. So are the lines of the tetrahedron exept OY and YZ.

If a and b are positive the map can be made continuous.

$$P^t = \begin{pmatrix} 1 & vt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a^t & 0 \\ 0 & 0 & 0 & b^t \end{pmatrix}$$

OXZ contains a set of general pathcurves like in figure **??**. OXY and ∞ each contain a set of logarithmic pathcurves like in figure **??**. All other pathcurves are twisted ones, each within one of *four* regions. Compare with section 3.2.6, where there are only two regions. The shape has two degrees of freedom; it is determined by $\frac{\ln a}{v}$ and $\frac{\ln b}{v}$; v is the speed.

- If 1 is between a and b, say $0 < a < 1 < b$, (and $v > 0$) then the common source is Z, the sink O, limiting directions are ZX and OX, limiting osculating planes ∞ and OXY. See figure **??**.

- In the other case, say $1 < a < b$, (and $v > 0$) each curve originates at X in direction XY and terminates in O, direction ZO; the limiting osculating planes are ∞ and OXZ. See figure **??**.

Note the difference between these two cases. In the former the limiting tangents meet in X, their join is OXZ. In the latter they are skew.

At first sight the curves look very much the same as the conic turns. The difference becomes obvious if we compare sets of curves, each building a surface, see figures **??**, **??** and **??**. In the first figure the surface is a semi-cone. In the other two it is a cone-like shape with a triangular opening at the rear side. In the first figure the (bottom) border is a conic, in the other two a (non-closed) plane logarithm. Also, the limiting osculating planes of the conic turn conincide.

3.3.3 Plane spirals

$$P = M \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} M^{-1} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}, \quad v \in \langle -\pi, \pi \rangle \setminus \{0\}\,,\; r > 0$$

see 1.4.2. X, Y, OYZ and OXZ are invariant; XY is point-, OZ planewise invariant.

The map can be made continuous.

$$P^t = \begin{pmatrix} e^{svt} & 0 & 0 & 0 \\ 0 & e^{svt} & 0 & 0 \\ 0 & 0 & \cos vt & -\sin vt \\ 0 & 0 & \sin vt & \cos vt \end{pmatrix}$$

where $s = \frac{\ln r}{v}$ is the shape. Each invariant plane contains a set of

- concentric conics if $s = 0$ (see figure **??**);
- spirals if $s \neq 0$ (see figure **??**).

There are no twisted curves.

3.3.4 Long spirals

$$P = M \begin{pmatrix} 1 & w' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} M^{-1} = \begin{pmatrix} 1 & w & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r\cos v & -r\sin v \\ 0 & 0 & r\sin v & r\cos v \end{pmatrix}, \quad v \in \langle -\pi, \pi \rangle \setminus \{0\}, \; r > 0, w \neq 0$$

X, XY, OZ and OXZ are invariant. The map can be made continuous.

$$P^t = \begin{pmatrix} 1 & wt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^t \cos vt & -r^t \sin vt \\ 0 & 0 & r^t \sin vt & r^t \cos vt \end{pmatrix}$$

OZ contains an elliptic, XY a parabolic motion. There is only one region. We can take v as the speed of the system. The shape is determined by $(\ln r)/v$ and w/v.

- If $r = 1$, the invariant plane contains a bundle of concentric conics like in figure **??**. The remaining pathcurves are spirals on cones. Each curve originates and terminates in X, the common apex of all cones. See figure **??**. The shape has one degree of freedom. In the affine case we have the *helix*.
- If $r \neq 1$, the invariant plane contains a set of spirals like in figure **??**. The remaining pathcurves are spirals on 'vortices', all originating in X. For $t \to \infty$ each orbit stretches and approaches OZ and OXZ. The vortices have their source in their covering plane and extend through the whole space. They are 'twice as long' as the vortices of §3.4.2. See figure **??**. The shape has two degrees of freedom.

3.4 Four different eigenvalues

3.4.1 All real: tetrahedral system

$$P = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \notin \{0, 1\}$, no two of them equal.
The fundamental tetrahedron is invariant.

If a, b and c are positive the map can be made continuous.

$$P^t = \begin{pmatrix} a^t & 0 & 0 & 0 \\ 0 & b^t & 0 & 0 \\ 0 & 0 & c^t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the general real type, or the *tetrahedral* type. Each line of the tetrahedron contains a hyperbolic motion, Each invariant plane contains a triangular system like in figure **??**. We can take $v = \ln a$ as the speed. The shape is determined by $\frac{\ln b}{v}$ and $\frac{\ln c}{v}$, so it has two degrees of freedom. If say $a > b > c > 1$, then O is the source of each twisted pathcurve, X their common sink. XY and OZ are tangent to, OYZ and ∞ are osculating planes of each twisted curve. Each curve is within one of eight regions. See figure **??**.

Two orbits in 'opposite' regions add up to a closed curve. Depending on the point of view (centre), the projection of this curve is topologically equivalent to

- a bow or
- an oval or
- a lemniscate.

The last one we call a *path lemniscate*. It is different from the lemniscates of Lissajous/Gerono, Booth and Bernoulli.

3.4.2 Two real, two imaginary

$$P = M \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix} M^{-1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}, \quad a, b \neq 0, a \neq b$$

X, Y, OYZ, OXZ, XY and OZ are invariant.

If a and b are positive the map can be made continuous.

$$P^t = \begin{pmatrix} a^t & 0 & 0 & 0 \\ 0 & b^t & 0 & 0 \\ 0 & 0 & \cos vt & -\sin vt \\ 0 & 0 & \sin vt & \cos vt \end{pmatrix}$$

OZ contains an elliptic motion, XY a hyperbolic one. The shape is determined by $\frac{\ln a}{v}$ and $\frac{\ln b}{v}$, v is the speed. There are two regions. We have the following cases.

- 1 is between a and b, e.g. $0 < a < 1 < b$.
  Each invariant plane contains a set of spirals like in figure **??**. In one plane they are outward, in the other inward. The remaining pathcurves are spirals on 'eggs', see figure **??**. They all originate in X and terminate in Y, touching OXZ resp. OYZ. The shape has two degrees of freedom.

- One of a, b equals 1, e.g. $a = 1, b > 1$.
  One invariant plane contains spirals, the other concentric conics, like figure **??**. The remaining pathcurves are spirals on semi-cones with common source Y. For $t \to \infty$ they approach a conic in OXZ. See figure **??**. The shape has one degree of freedom. Note the difference with the helix of §3.3.4.

- a, b at one side of 1, e.g. $a < b < 1$.
  Again the invarant planes contain spirals, but now all inward. The remaining pathcurves are spirals on 'vortices' with common source X, and common 'covering plane' OYZ. For $t \to \infty$ the orbits stretch towards OZ. See figure **??**. The shape has two degrees of freedom.

3.4.3 All imaginary

$$P = M \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \overline{\mu} \end{pmatrix} M^{-1} = \begin{pmatrix} r\cos w & -r\sin w & 0 & 0 \\ r\sin w & r\cos w & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{pmatrix}$$

where $r > 0$ and $v, w \in \langle -\pi, \pi \rangle \setminus \{0\}$.

$$P^t = \begin{pmatrix} r^t\cos svt & -r^t\sin svt & 0 & 0 \\ r^t\sin svt & r^t\cos svt & 0 & 0 \\ 0 & 0 & \cos vt & -\sin vt \\ 0 & 0 & \sin vt & \cos vt \end{pmatrix}$$

The lines XY and OZ are invariant, each containing an elliptic motion. The shape is determined by $(\ln r)/v$ and $s = w/v$; v is the speed of the system. There are three cases.

- $r \neq 1$. Every point not on one of the invariant lines, winds infinitely through space. With positive and increasing t the orbit stretches and approaches the line OZ while winding around it. If $t << 0$ and decreasing, it stretches again, but now coming closer and closer to XY while winding around it. These two windings however are different in quality. If say $r > 1$ and $|v| < |w|$, then for $t \to \infty$ the orbit really winds around XY like a coil. At the other hand, for $t \to -\infty$ the line OZ is covered by almost straight lines that turn around it while coming closer and closer. They resemble a one-blade hyperboloid; however this surface itself winds around OZ as well as around XY, becoming more and more narrow and cylindric[6]. Its equation is of the form

$$x_0^2 + x_1^2 = r^{\frac{2}{v}\arctan\frac{x_2}{x_3}+k\pi}(x_2^2 + x_3^2), \quad k \in \mathbf{Z}$$

A curve of this type we'll call a *line winding* (Ge: *Geradenwickel*; Fr: *enroulage de lignes*; Du: *lijnenwikkel*). See figure **??**. The shape has two degrees of freedom.

- If $r = 1$ there is no such stretching for $t \to \pm\infty$. Exept for the situation of the third case (see below) the points stray through space without any goal, which is why we call a curve of this type a *stray* (Ge: *Irrkurve*; Fr: *courbe errante*; Du: *dwaalcurve*). Now each orbit is on a hyperboloid. The orbit is *dense* in the surface, like the rationals being dense in the reals. See figure **??**. The shape has one degree of freedom.

- If in addition to $r = 1$ there exist integers n, m such that $mv = nw$ i.e. $s = m/n$ is rational, then the curves are closed. We call a curve of this type a *multi loop* (Ge: *Multi-Schleife*; Fr: *multi-boucle*; Du: *multi-lus*). See figure **??**. There are countably infinite shapes.

[6]Edwards [3] calls it the *Chalice surface*.

4 Summary

Below we summarise the results of the previous sections. The first column contains the dimension of the space; the second the number of roots of the characteristic polynomial. The third refers to the subsection where the case is treated. If the map can be made continuous for all values of its paramaters, then the column 'cont' contains a 'y'. The last but one contains the degree of freedom of the shape. The last one shows the 'dimension' of the system, i.e. 3 if the system has twisted curves, 2 if it has not but does have non-straight plane curves, etc.

n	roots of char. pol.	Eigen-space structure	cont	name	shape d.o.f.	system-dim
1	1 real	1.2	y	parabolic motion	0	1
	2 real	1.3	n	hyperbolic motion	0	1
	2 im	1.4	y	elliptic motion	0	1
2	1 real	2.1.1	y	elation	0	1
		2.1.2	y	contangential conics	0	2
	2 real	2.2.1	n	homology	0	1
		2.2.2	n	logarithms	1	2
	3 real	2.3.1	n	triangular system	1	2
	1 real, 2 im	2.3.2	y	spirals	1	2
			y	concentric conics	0	2
3	1 real	3.1.1	y	elation	0	1
		3.1.2	y	parabolic congruence	0	1
		3.1.3	y	contangential conics	0	2
		3.1.4	y	cubics	0	3
	2 real (1,3)	3.2.1	n	homology	0	1
		3.2.2	n	plane logarithms I	1	2
		3.2.3	n	conic turns	1	3
	2 real (2,2)	3.2.4	n	hyperbolic congruence	0	1
		3.2.5	n	plane logarithms II	1	2
		3.2.6	n	long logarithms	1	3
	2 im (2,2)	3.2.7	y	elliptic congruence	0	1
		3.2.9	y	sheafs	1	3
	3 real	3.3.1	n	triangular systems	1	2
		3.3.2	n	short logarithms	2	3
			n	skew short logarithms	2	3
	1 real, 2 im	3.3.3	y	plane spirals	1	2
			y	concentric conics	0	2
		3.3.4	y	long vortex spirals	2	3
			y	helixes	1	3
	4 real	3.4.1	n	tetrahedral system	2	3
	2 real, 2 im	3.4.2	n	egg spirals	2	3
			n	cone spirals	1	3
			n	vortex spirals	2	3
	4 im	3.4.3	y	line windings	2	3
			y	strays	1	3
			y	multi loops	0[7]	3

[7] There are countably infinite shapes.

5 Metamorphosis and degeneration

There are many ways of transforming one pathcurve system into another. Sometimes we get a *degeneration*: the homology is a degenerate spiral system (with zero angular speed). In other cases we prefer the term *metamorphosis*, for instance the transistion of the triangular, via the logarithmic to the spiral type. To be precise, a transformation of one pathcurve system into another one with a different eigenspace structure is called a *metamorphosis* if each system has $n+1$ different invariant points, n being the dimension of the projective space.

We present a summary of the most important transformations.

5.1 Planar systems

5.1.1 Metamorphosis of triangular into spiral system

We start with the **triangular type**, fig. 5. If we look at the sides of the triangle we note that

- from the source O there are 4 outgoing motions
- at the stationary point Y there are 2 out- and 2 ingoing motions
- at the sink X there are 4 ingoing motions.

If we move the stationary point Y of the system onto the source O (or the sink), we get the **logarithmic type**, fig. 4. One line has a parabolic motion, the other still a hyperbolic one. The spiral type is about to come into existence.

From the matrix representation one might expect to get the homology of section 2.2.1. However, considering that

$$\begin{pmatrix} p & \frac{1}{b-a} \\ 0 & \frac{1}{p} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \frac{1}{p} & \frac{1}{a-b} \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & p \\ 0 & b \end{pmatrix}$$

and that the last matrix is more general than its conjugate in the middle at the left side, we must accept that in general the triangular system transforms into the logarithm.

If we split the coinciding points O, Y into a pair of complex conjugates, we get the **spiral type**. There remains one real invariant point and one invariant line with an elliptic motion.

5.1.2 Degenerations

If we decrease the angular speed of the spirals to zero (or increase the absolute value of the radial speed to infinity), we get the **homology**, fig. 3. If we move the invariant point of the homology onto the invariant line we get the **elation**, fig. 1.

If we decrease the absolute radial speed of the spirals to zero, we get the **concentric conics**, fig. 6. If we move the invariant point of the concentric conics onto the invariant line we get the **contangential conics** fig. 2.

There are several other ways of transforming one system into the other. For instance, decreasing the speed on one side of the triangle of the first system to zero leads to the homology again.

5.2 Space systems

For sake of brevity we restrict to a maximum of 4 invariant points, 6 invariant lines and 4 invariant planes, i.e. types that have twisted curves.

It should be noted that it is hardly possible to make drawings of the 3-dimensional metamorphoses, and yet it is indispensable to draw sketches in order to support the imagination.

5.2.1 Metamorphosis

1 We start with the **tetrahedral** type, section 3.4.1, fig. 19, which has 4/6/4 invariant points/lines/planes, all real. We suppose that $a > b > c > 1$. If we look at the sides of the tetrahedron we note that

  - from the source O there are 6 outgoing motions
  - at the sink X there are 6 ingoing motions
  - at one stationary point, Z, there are 4 out- and 2 ingoing motions
  - at the remaining (stationary) point, Y, there are 2 out- and 4 ingoing points

  From the tetrahedral type we can move one stationary point onto the other to get a **short logarithm**, section 3.3.2 fig. 13. Note that now $a > b = c > 1$, and that indeed we found the non-skew type. The same trick as in section 5.1.1 can be used to show that in general we do *not* get the plane triangular systems of section 3.3.1.

  Next we move the coinciding points into the imaginary, getting the **egg spiral**, section 3.4.2, fig. 20. The matrix becomes:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & r\cos v & -r\sin v & 0 \\ 0 & r\sin v & r\cos v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  with $a > r \approx b = c > 1$. Changing the shape-parameters (e.g. increasing r) transforms this one via cone spiral ($r = a$, fig. 21) into vortex spiral (fig. 22). Note that in these last three cases there are now two invariant (real) points, two invariant lines and two invariant planes.

  Going on from the egg-spiral, move O onto X, to get the **helix**, fig. 17.

  Finally, we also move O and X into the imaginary, getting one of the **chalice lines**, section 3.4.3, figs. 23-25 (try first to imagine the cone of the helix to transform into a hyperboloid, with a stray or multiloop on it). There are still two invariant lines, but no invariant real points, nor planes.

2 Again we start with the tetrahedral type, but now we move one of the stationary points towards source or sink, say Z onto O. We now get the short *skew* logarithm. Moving Z and O into the imaginary, results in the **short vortex spiral**. Next move X onto Y to get the **long vortex spiral**, and moving them into the imaginary finally leads to the line winding again.

3 After moving Z onto O, we can move Y onto Z, to get the **long logarithm**. The points and planes have multiplicity 2, $OX = YZ$ has multiplicity 4. Next we split one point in two complex conjugate ones to get a long spiral again, and finally split the other one to get at a chalice line. This indeed is rather artificial, but it is the only way to get the long logarithm.

We leave it as an exercise for the reader to check that moving Z onto X or Y onto O produces the plane logarithms of section 3.2.2, and that moving source O onto sink X produces the elation of section 3.1.1.

5.2.2 Degenerations

1 Let's start to move Z onto O, to get a short logarithm. From here we can also move Y onto $O = Z$. Now we seem to have 2 invariant points (multiplicity 1 and 3), 3 invariant lines (multiplicity 1, 2, 3) and 3 invariant planes (multiplicity 1, 2, 1). From duality consideartions this is not possible. So in moving Y to Z the 'single' line (the z-axis) turns towards XZ and we are left over with the **conic turn** of section 3.2.3, fig. 10. The two lines, the point $O = Y = Z$ and one plane all have multiplicity 3.

Finally we can move X onto the other. The planes collapse into one, as do the lines. We get the **cubic** of section 3.1.4, fig. 8. The single invariant point has multiplicity 4, as has the invariant plane. The invariant line has multiplicity 6.

Of course, one could transform the cubic by moving points into the imaginary. This, however, is too artificial to my taste.

2 We start from the long logarithm, fig. 11 (see the third metamorphosis of the previous section). Now instead of splitting one point into two imaginary ones, we move one point onto the other, getting the cubic again.

3 A last degeneration, again a rather artificial one, is produced by moving one invariant line of the line-winding onto the other, to get the **sheaf**. In doing so we should imagine the coil around one line (the part for t close to infinity) getting more and more narrow and finally collapsing onto its axis. The other part (for t near $-\infty$) transforms into the sheaf.

There are many ways of transforming one system into another by changing the shape-parameters. We leave this as an exercise for the reader.

5.2.3 Summary

We can visualize the above with the following scheme.