Symmetric partitioning of the projective line

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We want to put $n \geq 4$ points

$$A_0, A_1, A_2 \cdots A_{n-1}, A_n = A_0$$

on a projective line m in a 'symmetric' way. Since the only tool we have is the cross ratio, it is obvious that we require $(A_0A_1A_2A_3) = (A_1A_2A_3A_4) = \cdots = (A_{n-1}A_0A_1A_2)$.

Definition The points A_i generate a *symmetric partition* of m, or - shortly - the A_i are called *symmetric*, if the above condition is fulfilled.

Since $n \geq 4$ there exists a unique projective map on m that moves A_i to A_{i+1}. It is an elliptic motion, since there can't be any double points. All elliptic motions have the same shape[1], they differ only in the speed, which is one-to-one with the above cross ratio. This implies that if the lines m and m' both have n symmetric points A_i resp. A'_i there is a projectivity that maps A_i on A'_i for every i.

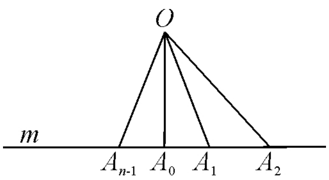

Figure 1: Symmetric partition

Now, how can we construct n symmetric points on m? Take a point O at unit lenth from m, assuming the last not at infinity. The rotation r on O about π/n - not $2\pi/n$! - of the pencil of lines through O is a projective map. Hence, taking A_0 on m arbitrary, or rather - for convenience - at minimal distance from O, the points $A_i = sr^is^{-1}A_0$ satisfy the required condition of the cross ratio, i.e. are symmetric on m.

What value has our cross ratio? Assuming OA_0 is perpendicular to m, $A_{n-1}A_0 = A_0A_1 = t = \tan\pi/n$ and $A_0A_2 = t_2 = \tan 2\pi/n = 2t/(1-t^2)$. Our cross ratio - in some suitable definition - equals

$$(A_{n-1}A_0A_1A_2) = \frac{A_{n-1}A_0}{A_1A_0} : \frac{A_{n-1}A_2}{A_1A_2} = \frac{t}{-t} : \frac{t_2+t}{t_2-t} = \frac{t^2+1}{t^2-3}$$

Note that this number is negative for all values of $n \geq 4$.

For $n = 4$ we have $\tan\pi/4 = 1$ and the cross ratio becomes -1, which means our points lie harmonic. Not really surprising.

[1]See my Classification of Pathcurves, MPK 219.

For $n = 5$ we have $\tan \pi/5 = \sqrt{5 - 2\sqrt{5}}$, hence our cross ratio becomes

$$\frac{1-\sqrt{5}}{2} = -0.618\ldots$$

which is the negative of the golden ratio. Already a bit surprising.

For $n = 6$ we have $t = \sqrt{3}/3$ and the cross ratio becomes $-1/2$. Is there a geometric explanation for this result? Yes there is. Take any triangle as coordinate triangle, and any admitted point as unit point. Draw the lines from the vertices of the triangle to the unit point. Now we have 6 lines. Cut these with an arbitrary seventh line to produce 6 symmetric points. Take any four consecutive of these and prove that their cross ratio is $-1/2$.

For $n = 7$ we have $t = .482$ and the cross ratio becomes -0.445[2].

For $n = 8$ we have $t = -1 + \sqrt{2} = 0.414$ and the cross ratio becomes $1 - \sqrt{2} = -0.414$.

If n increases to infinity, t vanishes and the cross ratio becomes $-1/3 = -0.333\ldots$. This *is* surprising, isn't it? Is there any geometric meaning in it? We look forward to any significant explanation.

n	π/n	t	*crossratio*	*approximation*
4	0.79	1.00	-1	
5	0.63	0.73	$(1-\sqrt{5})/2$	-0.618033988749895
6	0.52	0.58	$-1/2$	-0.5
7	0.45	0.48		-0.445041867912629
8	0.39	0.41	$1-\sqrt{2}$	-0.414213562373095
9	0.35	0.36		-0.394930843634698
10	0.31	0.32		-0.381966011250105
∞	0	0	$-1/3$	-0.333333333333333

[2] In this case t is an algebraic number too. To find its 'exact' value, one has to solve the equation $x^3 - 112x - 448 = 0$ which has discriminant $\sqrt{2 \times 10037}$. Hence, for $n = 7$ the cross ratio is not a 'nice' or simple number like the previous ones. Nor do we know of any geometric meaning.