Vector spaces and projective geometry

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October 29, 2017

Abstract A relation is established between the duality within projective geometry and the duality between a vector space and its dual space of linear funtions. The action of a linear function on a vector appears to be a cross ratio.

1 Introduction

Vector spaces are of fundamental importance, not only in Mathematics, but in many applied sciences as well, especially in Physics. Less familiar is the concept of the dual vector space - the space of linear functions on a vector space - though it is a basic concept in both Mathematics and Theoretical Physics. Projective spaces are - even among many mathematicians - alien or forgotten objects, that one encounters every once in a while, but hardly ever needs to use.

We will show in this article that projective spaces are at the heart of Linear Algebra, and that in particular the dual vector space gets a better understanding when seen in the light of projective geometry.

2 Prerequisites

The reader is supposed to be familiar with the basics of linear algebra and projective geometry, in particular with the concepts of *(number) field, vector space*, *projective space* and *cross ratio*.

In this article we will restrict to finite-dimensional vector spaces. Essentially these are the n-th powers of number fields: every n-dimensional vector space over a field F is isomorphic to the space F^n of n-tuples of numbers of F.

An n-dimensional projective space over F can be defined in various ways. One way is to extend Euclidean n-space with an $(n-1)$-dimensional hyperplane 'at infinity'. Another one is to 'divide' an $(n+1)$-dimensional vector space V by the relation $a \simeq \lambda a$, with $a \in V,\ \lambda \in F \setminus 0$. But we emphasize that it can be defined synthetically as well, that is, it can be constructed from a few relatively simple geometric axioms[1]. An element of F is also called a *scalar*, as opposed to the vectors of V.

[1]See e.g. my *On the Fundamentals of Geometry* available at www.mathart.nl .

3 Linear transformations

Let be given two vector spaces, V and W, over some field F. A *linear transformation* or *homomorphism* $f : V \to W$ is a map satisfying $f(v+w) = f(v)+f(w)$ and $f(av) = af(v)$ for all $v, w \in V,\ a \in F$. If V has dimension n and W has dimension m, then each homomorphism from V to W can be represented by an $m \times n$-matrix of scalars. This representation depends on the bases chosen in V and W: the i-th column of this matrix is the image of the i-th basis vector of V expressed in the basis of W.

One can add two maps $f, g : V \to W$ by defining $(f+g)v = f(v)+g(v)$. Scalar multiplication is defined by $(a.f)v = a.f(v)$. The zero transformation 0 maps all vectors of V on the zero vector of W. Thus, the linear transformations from V to W form a vector space $\mathrm{Lin}(V, W)$, of dimension $m.n$ over the same field F.

Fundamental concepts are kernel and image of a transformation. The kernel $\mathrm{Ker}\ f$ of a homomorphism $f : V \to W$ is the set of vectors of V that are mapped on the zero vector of W:

$$\mathrm{Ker}\ f \ = \ \{\ v \in V \mid f(v) = 0_W\ \}$$

The image of f is simply the set of images $f(v)$:

$$\mathrm{im} f \ = \ \{\ f(v) \mid v \in V\ \}$$

which may be less than W. Kernel and image are vector spaces of their own, *subspaces* of V resp. W. The sum of the dimensions of kernel and image of f is the dimension of V:

$$\dim(\ \mathrm{Ker}\ f) + \dim(\mathrm{im} f) = \dim(V)$$

In general, a linear transformation is not invertible (bijective), but if it is, it is called an *isomorphism*, and the spaces V and W are called *isomorphic*. In that case they necessarily have the same dimension. Isomorphisms are precisely those maps that have square matrices with non-zero determinant.

All n-dimensional vector spaces over some fixed field F are isomorphic, and in particular isomorphic to the set of n-tuples of elements of F, viz. F^n.

4 The dual space

Given a vector space V over a field F, its dual space, V^*, is defined as the space of linear functions $G : V \to F$. Since $F = F^1$ is a 1-dimensional vector space over F, $V^* = \mathrm{Lin}(V, F)$ is a vector space of the same dimension as V, and hence isomorphic to V. Its elements are called *covectors* or *dual vectors*[2].

Let $e_1 \dots e_n$ be a basis for V. For each vector v there is a unique n-tuple $(a_1, \dots a_n)$ such that $v = \sum_i a_i e_i$. The a_i are called he *coordinates* of v with respect to this basis. If $G : V \to F$ is a linear function, we have: $G(v) = G(\sum_i a_i e_i) = \sum_i a_i G(e_i)$. Hence, G is completely determined by the images of the vectors e_i.

[2] We will, with J.M. Lee ([Lee]), use the term covector

Now define the covectors $E_i : V \to F$ by $E_i(e_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. If we apply E_i to an arbitrary vector $v = \sum_j a_j e_j$, it picks out the i-th coordinate, i.e. $E_i(v) = a_i$. But then we have $G(v) = \sum_i a_i G(e_i) = \sum_i G(e_i)E_i(v)$, so we can write $G = \sum_i G(e_i)E_i$, or, putting $b_i = G(e_i)$, $G = \sum_i b_i E_i$. That is, for each $G \in V^*$ there is some n-tuple (b_i) such that $G = \sum_i b_i E_i$, so the E_i form a basis for V^*. It is called the dual basis or cobasis corresponding to (e_i).

In a similar way one can construct the dual V^{**} of V^*. This again is a vector space isomorphic to both V^* and V. However, it appeares that there is a canonical (=natural) isomorphism between V^{**} and V. Define $\iota : V \to V^{**}$ as follows. First, notice that $\iota(v)$ is an element of V^{**}, that is: a map from V^* to the number field F:

$$\iota(v) : V^* \to F$$

So for every $G \in V^*$ we have to define what $\iota(v)(G)$ means. What if we simply define $\iota(v)(G) = G(v)$? Can we prove that ι is an isomorphism? Obviously, it is linear, because every G is linear:

$$\iota(v + w)(G) = G(v + w) = G(v) + G(w) = \iota(v)(G) + \iota(w)(G)$$

for every $G \in V^*$, hence $\iota(v + w) = \iota(v) + \iota(w)$. Similarly $\iota(av) = a\iota(v)$.
Next, suppose $\iota(v) = \iota(w)$, i.e. $G(v) = G(w)$ for all G. In particular, given bases (e_i) and (E_i) as above, $E_i(v) = E_i(w)$ for all i. That is, v and w have the same coordinates and must be equal. But then ι is injective (one-one), and since V and V^{**} have the same dimension, it is bijective. So ι is indeed an isomorphism.

Because of this 'naturality' it would be pedantic to distinguish between V^{**} and V, leave alone to consider V^{***} etc. Yet there is a fundamental difference in quality: a vector is definitely to be distinguished from a function, so how should a function on a function space become a vector again? That seems hardly possible as long as these functions themselves have no geometrical meaning.

5 The cross ratio

Given three distinct points P, Q, R, and a fourth one S, all on one projective line. The cross ratio $(PQRS)$ is defined as the homogeneous coordinates $(\mu : \lambda)$ of S with respect to the system of reference $P(1 : 0)$, $Q(0 : 1)$, $R(1 : 1)$.

We will show that this definition is in accordance with the ususal Euclidean one. Let A, B, C, D be points on a Euclidean line, in the order $ABCD$ and with $AB = \alpha > 0$, $BC = \beta > 0$ and $CD = \gamma > 0$ then the Euclidean definition gives

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{(\alpha + \beta)(\beta + \gamma)}{\beta(\alpha + \beta + \gamma)}$$

We can give the following homogeneous coordinates to our points: $A(0 : 1)$, $B(\alpha : 1)$, $C(\alpha + \beta : 1)$, $D(\alpha + \beta + \gamma : 1)$ to our points. Now apply the projective map

$$f = \begin{pmatrix} 1 & -\alpha \\ \beta/(\alpha + \beta) & 0 \end{pmatrix}$$

Verify that $f(A) = (1:0)$, $f(B) = (0:1)$, $f(C) = (1:1)$ and $f(D) = (\frac{(\alpha+\beta)(\beta+\gamma)}{\beta(\alpha+\beta+\gamma)}:1)$ indeed.

It is possible to extend the definition to the cases $(AABB)$, $(ABAB)$ and $(ABBA)$ but we will have no need for this.

6 The 2 dimensional case

In this section we will use capitals for lines and covectors, and lower case type for points and vectors.

Let be given the 2-dimensional vector space $\mathcal{V}$ over the reals, with a fixed basis e_1, e_2 and zero vector o.

Let $\mathcal{V}^*$ be the dual vector space, i.e. the space of *linear* functions or covectors $G : \mathcal{V} \to \mathbf{R}$. A natural basis for the dual space is E_1, E_2 defined as follows: $E_i(e_j) = \delta_{ij}$, the Kronecker delta.

Every vector a can be written as

$$a = \Sigma \alpha_i e_i = (\alpha_1, \alpha_2)$$

and every linear function B as

$$B = \Sigma \beta_i E_i = [\beta_1, \beta_2]$$

In particular $o = (0,0)$ and $O = [0,0]$, the zero function that maps each vector onto the number 0.

Now $B(a) = \Sigma \alpha_i \beta_i = a(B)$, the last equality because of the duality between $\mathcal{V}$ and $\mathcal{V}^*$: vectors behave as linear functions on $\mathcal{V}^*$. Though this looks very much like a scalar product of vectors, it is not, since it is the mutual action of a vector and a covector. It has no relation to any metric on $\mathcal{V}$, nor on $\mathcal{V}^*$. In addition it is heavily dependent of the basis, whereas a metric is not.

The duality between $\mathcal{V}^*$ and $\mathcal{V}$ reminds us of the duality inside projective spaces, viz. between points and lines in the plane. Is this a coincidence, or is there a connection between these two dualities?

We will set up a correspondence between the linear functions and the lines.

6.1 Example

Let's first have a closer look at the kernel of a linear function. Consider the linear function $G(x,y) = 2x - 3y \in \mathcal{V}^*$. Its kernel is the line $2x - 3y = 0$, see left figure. Now the map $H = 2G$ has the same kernel as G. So it is not possible to set up a 1-1-correspondence between kernels and covectors.

If we take *level sets* (or *isolines* or *contour lines*) in stead of kernels the situation is better. A *level set* is the inverse image of a number. Let G again be the linear function $2x - 3y$. The 5-level is the set

$$G^{-1}(5) \;=\; \{\ (x,y) \in \mathbf{R}^2 \mid 2x - 3y = 5\ \}$$

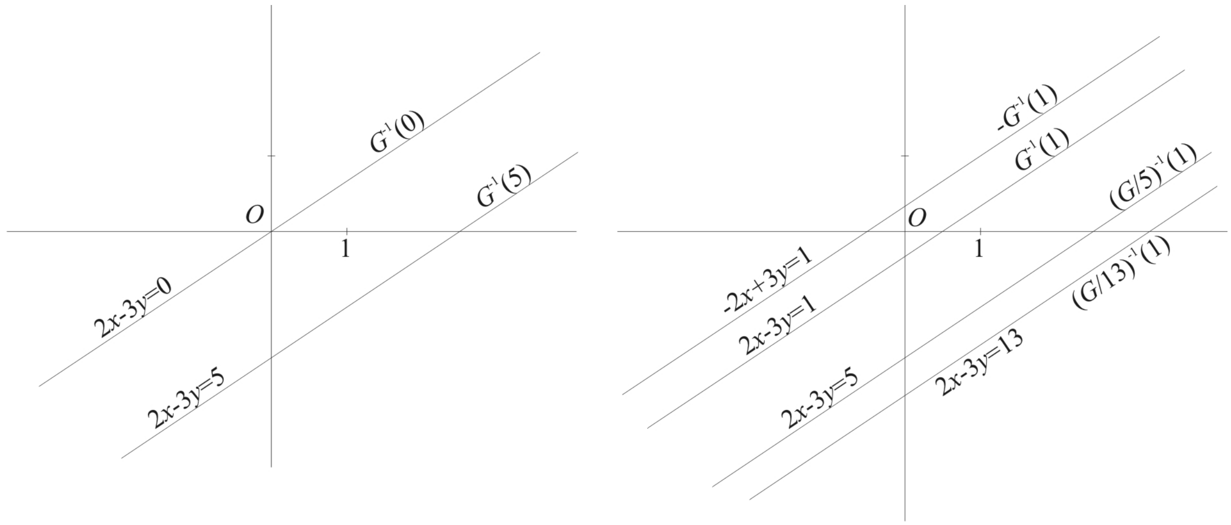


Figure 1: kernel and level sets

that is, a line parallel to the above kernel, which is also shown in the left figure.

Consider all the level sets of some constant γ, while varying our G. Since the value of γ is not important (that is, with any fixed value $\neq 0$ we can do the next considerations), we consider the level sets of $\gamma = 1$.

Now, whereas $H = 2G$ and G have the same kernel, they have *not* the same level-1 set. The level-1 set of G, $2x - 3y = 1$, is not the same as $4x - 6y = 1$, the level-1 set of H. In fact, the last one, which can be written $2x - 3y = 1/2$, is closer to the origin than the first. This last observation, can be reformulated as: *the bigger λ ($\lambda > 0$), the closer the level-1 set of λG is to the origin*, or *the smaller λ (but always positive) , the farther its level-1 set from the origin*. In the right figure the level-1 sets of $-G, G, G/5$ and $G/13$ are shown.

If λ becomes infinitely small, i.e. G is the zero-map, its level set is empty. But, if we embed our vector space $\mathbf{R}^2$ in a projective space, by adding, in the usual way, a line at infinity, ∞, then we find that, independent of G, we have

$$\lim_{\lambda \to 0} (\lambda G)^{-1}(1) = \infty$$

as long as $G \neq 0$. So it makes sense to redefine level sets in the following way:

- if $G \neq 0$ then $G^{-1}(1) \ = \ \{ \ v \in V \mid G(v) = 1 \ \}$
- $0^{-1}(1) = \infty$

While $G(0, 0) = 0 \neq 1$ for each G, no level set contains o, just like no vector is in ∞.

This gives a perfect relation between vector spaces and projective geometry. In fact, from a geometrical point of view, it is natural to consider a vector space as a subset of a projective space.

6.2 To make a projective space from a vector space

Returning to the general 2-dimensional case, the first thing we have to do in order to connect lines and covectors is to turn this vector space into a projective space $\mathcal{S}$. Add points at infinity to $\mathcal{V}$ in the usual way, to get the set of all points $\mathcal{S}_0$ of $\mathcal{S}$. Add the line at infinity ∞ to the lines of $\mathcal{V}$ to get the collection $\mathcal{S}_1$ of all lines of $\mathcal{S}$. Define $a_1(1:0:0)$ as the meeting point of oe_1 with

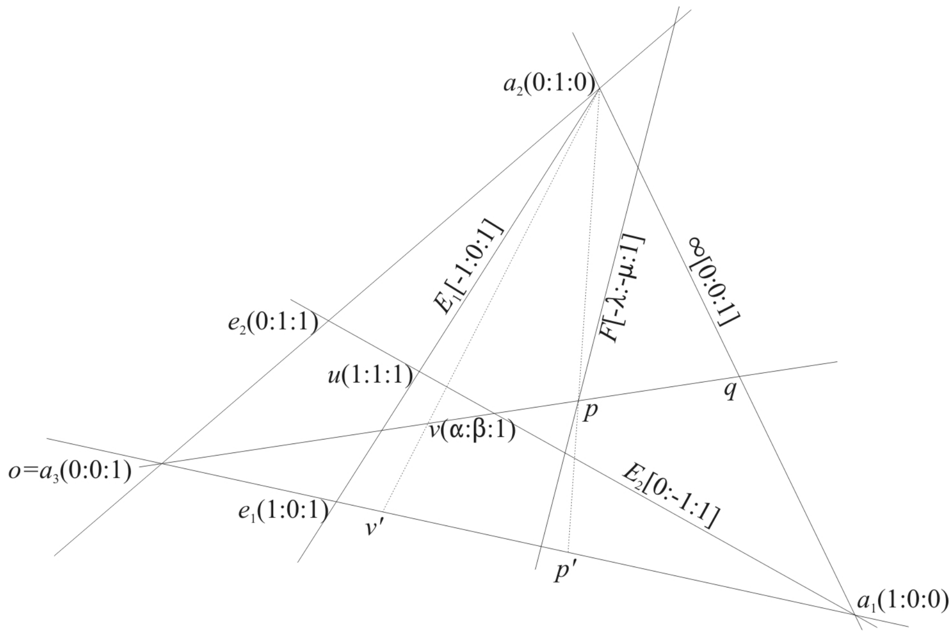


Figure 2: a vector space, its dual and the projective space

∞, $a_2(0:1:0)$ as the meeting point of oe_2 with ∞, and $a_3 = o = (0:0:1)$, the zero vector. Let $u = e_1 + e_2 = (1:1:1)$ be the meeting point of the lines a_1e_2 and a_2e_1. Now $a_1,\ a_2,\ a_3,\ u$ is a system of reference for S_0. Observe that if $v(\alpha,\beta) = (\alpha:\beta:1), w(\gamma,\delta) = (\gamma:\delta:1)$ are vectors, their sum is $v + w = (\alpha+\gamma:\beta+\delta:1)$. And $\lambda v = (\lambda\alpha:\lambda\beta:1)$.

Define $A_1 = a_2a_3 = [1:0:0]$, $A_2 = a_1a_3 = [0:1:0]$, $A_3 = \infty = a_1a_2 = [0:0:1]$ and the unit line $U = [1:1:1]$ as the line through $-e_1$ and $-e_2$. Now $A_1,\ A_2,\ A_3,\ L$ is a system of reference for $\mathcal{S}_1$.

We recall that with these bases the cross product of two distinct poins is their connecting line, and the cross product of two distinct lines is their meeting point.

6.3 Isolines

We will consider level sets (or contour lines or isolines): lines that connect points (vectors) with equal function values, viz. function value 1 (one could, however, take any other fixed non-zero number again). The level-1 set of the basis covector E_1 is

$$E_1^{-1}(1) = \{v = \lambda e_1 + \mu e_2 | E_1(v) = 1\} = \{e_1 + \mu e_2\} = e_1a_2 = [-1:0:1]$$

and the level 1 set of E_2 is $e_2a_1 = [0 : -1 : 1]$. So, if the normal coordinates of a covector G are $[\alpha, \beta]$ then its homogeneous coordinates are $[-\alpha : -\beta : 1]$.

Let $G = \lambda E_1 + \mu E_2 = [-\lambda : -\mu : 1]$ be an arbitrary covector. We will compute $G^{-1}(1)$. Suppose first $\mu \neq 0$ and let $v = \alpha e_1 + \beta e_2$ such that $G(v) = 1$. Then $G(v) = \alpha\lambda + \mu\beta = 1$, which implies $\beta = (1 - \alpha\lambda)/\mu$ or

$$v = \alpha e_1 + (1 - \alpha\lambda)e_2/\mu = \alpha e_1 + e_2/\mu - \alpha\lambda e_2/\mu = e_2/\mu + \alpha(e_1 - \lambda e_2/\mu)$$

Hence $v = (\alpha : (1/\mu - \alpha\lambda/\mu : 1))$ and $G^{-1}(1) = [-\lambda : -\mu : 1]$ indeed. Verify that this last relation also holds if $\mu = 0$.

The level 1 line of $[-\lambda : -\mu : 1]$ meets the axes in the points $(1/\lambda : 0 : 1)$ and $(0 : 1/\mu : 1)$. If we move either or both λ and μ to infinty, we get contour lines through the point $o(0 : 0 : 1)$. So this point has the same unreachable status with respect to the covectors as the line at infinty has with respect to the vectors. At the other hand, moving either or both λ and μ to 0 we get the line at infinity, so we *define* the level 1 contour line of the zero function as the line at infinity.

6.4 The mutual action of a vector and a covector

Let be given a vector $v = (\alpha, \beta) = (\alpha : \beta : 1)$ and a covector $G = [\lambda, \mu] = [-\lambda : -\mu : 1]$. Let L be the line through o and v, i.e. $L = [-\beta : \alpha : 0]$. The meeting point of L and G is $p(\alpha : \beta : -\alpha\lambda - \beta\mu)$, and that of L and ∞ is $q(\alpha : \beta : 0)$. Next apply the regular projective map (coordinate transformation) with matrix

$$\frac{1}{\alpha}\begin{pmatrix} 1 & 0 & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

then the images or our four points are

$$o' = o = (0 : 0 : 1),\ v' = (1 : 0 : 1),\ p' = (1 : 0 : \alpha\lambda + \beta\mu),\ q' = (1 : 0 : 0)$$

or, dropping the middle coordinate

$$o' = o = (0 : 1),\ q' = (1 : 0),\ v' = (1 : 1),\ p' = (1 : \alpha\lambda + \beta\mu)$$

Hence the cross ratio $(oqvp) = (oq'v'p') = \alpha\lambda + \beta\mu = G(v)$. This cross ratio equals $(OQVP)$ where O, Q, V and P are the lines connecting o, q, v and p respectively with any point not on L, in particular the meeting point of G and ∞: here too we have full duality. We can also write this cross ratio as $(o\infty vG) = (\infty oGv)$, where the two lines (covectors) separate the two points (vectors).

Note that this cross ratio is independent of the chosen basis e_1, e_2, but is does depend on the choice of o and ∞.

The mutual action of a vector and a covector is a cross ratio. This cross ratio does not depend on the bases, but only on the choice of o and ∞.

7 The general case

It does not require much imagination to see that the above can be extended to any dimension.

Let be given a n-dimensional projective space $\mathcal{S}$, with set of points $\mathcal{S}_0$ and set of hyperplanes $\mathcal{S}_{n-1}$. Take a fixed point $o = a_0$ and a fixed hyperplane $A_0 = \infty$ not containing o. Let G be any plane not containing o, and v be any point not in ∞. Then we *define* the mutual action of them as $G(v) = v(G) = (\infty o G v) = (qopv)$, where p is the meeting point of line ov with G and q is the meeting point of ov with ∞. Observe that in this cross ratio the two points separate the two hyperplanes.

Take points $a_1, a_2, \ldots, a_n \prec A_0$ and a point $u \not\prec A_0$ such that $a_0, \ldots, a_n, u$ form a system of reference for $\mathcal{S}_0$.

The hyperplane generated by all the a_k exept a_i is called A_i. The unit hyperplane $U[1:1:\ldots:1]$ is the plane through the points that have one coordinate 1, one coordinate -1 and all others 0. Then the A_i and U form a system of reference for $\mathcal{S}_{n-1}$.

Let $\mathcal{V} = \mathcal{S}_0 \setminus \{v \in \infty\}$ be the set of points not in ∞ and let $\mathcal{V}^*$ be the set of hyperplanes not containing o. For $0 < k \leq n$ let E_k be the hyperplane containing u and all a_i exept a_0 and a_k. Let e_k be the meeting point of line oa_k and E_k. The the e_k are a basis for $\mathcal{V}$ and the the E_k are a basis for $\mathcal{V}^*$.

Proposition 7.1 *Take any point $v(1:\alpha_1:\ldots:\alpha_n)$ (hence not in ∞) and one hyperplane $G[1:-\beta_1:\ldots:-\beta_n]$ (not containing o). Then $G(v) = v(G) = (\infty o G v) = (o \infty v G) = \Sigma_{i=1}^{n} \alpha_i \beta_i$.*

Proof. The proof is essentially the same as that in section 6.4. Let L be the line through o and v, $p(\Sigma\alpha_i\beta_i : \alpha_1 : \ldots : \alpha_n)$ the meeting point of L and G, and $q(0:\alpha_1:\ldots:\alpha_n)$ the meeting point of L and ∞. Consider the coordinate transformations

$$T = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & \alpha_1 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \alpha_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & \alpha_n \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} \alpha_n & 0 & 0 & \ldots & 0 & 0 \\ 0 & \alpha_n & 0 & \ldots & 0 & -\alpha_1 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & \alpha_n & -\alpha_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} / \alpha_n$$

They leave all basis points invariant exept the last one. T moves $(0:\ldots:0:1)$ to q and T^{-1} moves it back. Both maps leave ∞ invariant (although not pointwise). The images of our four points of L under T^{-1} are $o' = o$, $v' = (1:0:\ldots:0:1)$, $p' = (\Sigma\beta_i\alpha_i : 0 : \ldots : 0 : 1)$ and $q' = (0:\ldots:0:1)$. If we drop the middle coordinates we get $o = (1:0)$, $v' = (1:1)$, $p' = (\Sigma\beta_i\alpha_i : 1)$ and $q' = (0:1)$. Hence $G(v) = v(G) = (\infty o G v) = (qopv) = (oq'v'p') = \Sigma_{i=1}^{n}\alpha_i\beta_i$. $\diamond$

Let v be a point on G, that is $0 = Gv^\tau = 1 - \Sigma\alpha_i\beta_i = 1 - G(v)$ hence $G(v) = 1$. Clearly G is the level 1 hyperplane of the covector G.

In thit last sentence we encounter the matrix product Gv^τ. These matrices are determined up to a scalar factor, viz. $Gv^\tau = (\lambda G)v^\tau = G(\lambda v)^\tau = \lambda(Gv^\tau)$ for each $\lambda \neq 0$, hence can assume any value. So, in general this product can only be used for distinguishing 0 and $\neq 0$, i.e. determining incicence. **That it plays the above important role, is because we enforced affine coondinates by taking the first ones equal to 1.**

8 Computing with covectors

In the rest of this article points are named by capitals, lines by lower case type.

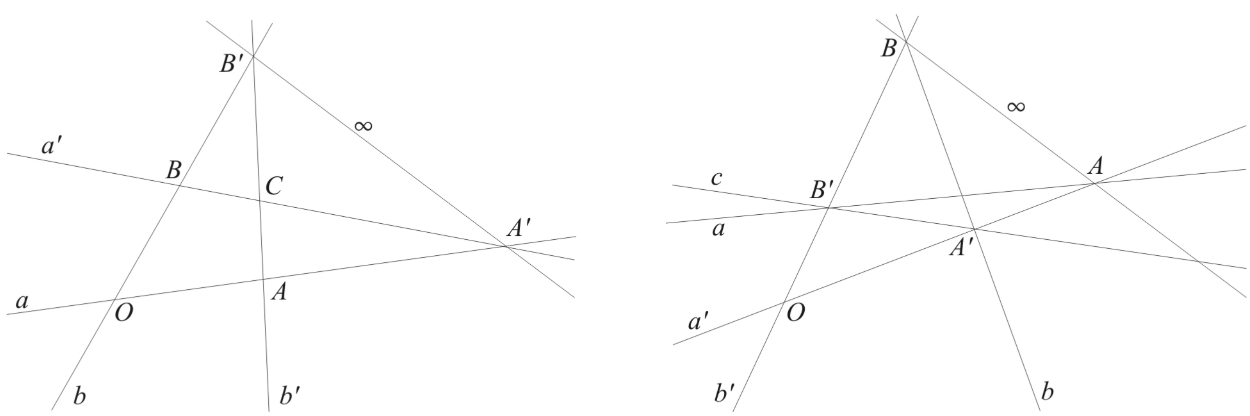
8.1 Adding covectors

The question arises: what is the geometrical meaning of adding two covectors? We will show this by an example in the projective plane. But first, let's recall what addition of vectors (represented by points) means. The vector $C = A + B$ is constructed as follows.

1 Join A and O by a line a, and B and O by b.

2 Next, draw a line $a' \parallel a$ through B, or - in projective terminology - draw a' through B and the meeting point A' of a and ∞. Similarly, draw b' through A and the meeting point B' of b and ∞.

3 Now the meeting point of a' and b' is C, the sum of A and B.



Addition of vectors... ...and co-vectors

We will carefully dualize this, in order to find the sum of the covectors represented by two lines a, b.

1 First let A and B be the meeting points of a resp. b with ∞.

2 Join A resp. B with O to get the lines a' resp. b'. A' is the meeting point of a' and b, B' is the meeting point of a and b'.

3 Now the line c joining A' and B' is the sum of a and b.

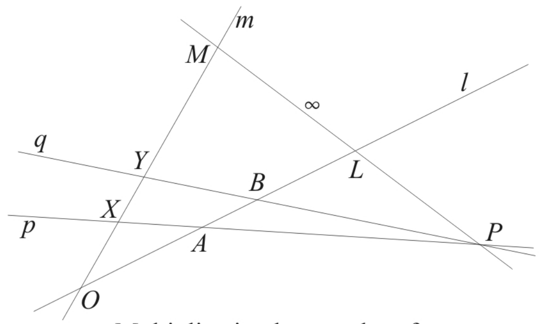
We leave it as an exercise to define $a+b$ if a, b and ∞ are concurrent (have a common point). You should also check that, indeed, if you add for instance the functions $\phi(x, y) = x - 3y$ and $\psi = 2x + y$ you get the function $(\phi + \psi)(x, y) = 3x - 2y$.

8.2 Scalar multiplication

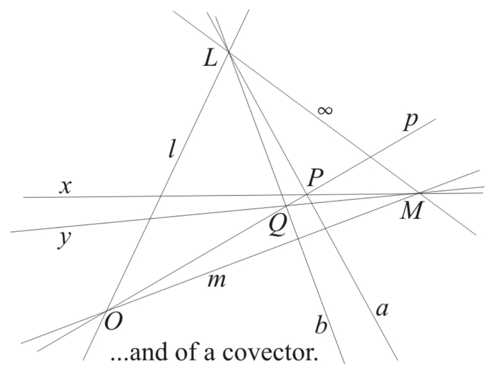
Next, we want to have a geometric interpretation of the product of a covector and a scalar.

Let's again restrict to the real projective plane. As we saw above, as soon as we have singled out a point O and a line ∞ we obtain the additive groups of $\mathbf{R}^2$ and its dual. But then we can construct the vectors $2A = A + A$, $3A$ etc. as well. Using a harmonic net[3] we can even find the rational multiples of vectors, hence approach every real multiple as closely as wanted.

So, given two collinear vectors, A and B, there is a unique real number λ such that $B = \lambda A$. Now, how do we find $Y = \lambda X$ for an arbitrary vector X? Well, this is - again - very similar to the Euclidean case.



Multiplication by a scalar of a vector...

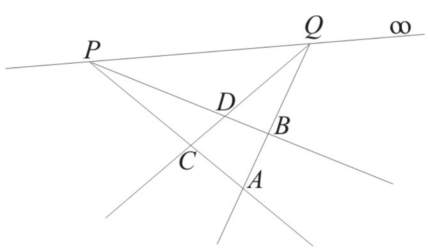


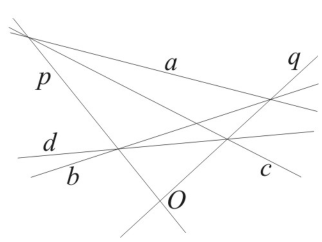
...and of a covector.

Let $m = OX$, $p = AX$, P the meeting point of p and ∞, $q = BP$, Y the meeting point of q and m. Then $Y = \lambda X$ is the desired point. (To define multiplication on AB one has to use an intermediate line.)

Dualizing, we proceed as follows. Let a, b and ∞ be concurrent, viz. share the point L. There is a number λ such that $b = \lambda a$. We want to construct $y = \lambda x$ for some given line x not through P. Let M be the meeting point of x and ∞ and P the meeting point of a and x. Draw the line $p = OP$ and let Q be its meeting point with b. Then $y = MQ$ is the desired line.

9 Free covectors



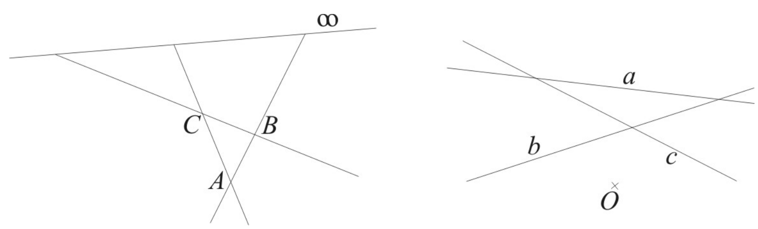


In this section we again restrict to dimension 2. A *free vector* is an equivalence class of the relation $(A, B) \sim (C, D) \quad \Leftrightarrow$ there are two points E, F such that $ABFE$ and $CDFE$ are

[3]See for instance Frank Ayres, Projective Geometry, 1967 McGraw-Hill, Schaum's ouline series

parallelograms. And four non-collinear finite points $ABCD$ are a parallelogram if AB and CD meet in a point at infinity and so do AD and BC. There is a one-one relation between vectors and free vectors: in each class there is exactly one with first element O.

Dualizing this we define on the set of (pairs of) lines not containing O the relation $(a, b) \sim (c, d) \quad \Leftrightarrow$ there are two lines e, f such that $abfe$ and $cdfe$ are dual parallelograms. A dual parallelogram then is a set of four distinct non-concurrent lines $abcd$, none containing O, such that the meeting point of a and b and the meeting point of c and d are collinear with O, and also the meeting point of a and d and the meeting point of b and c are collinear with O.



How does the relation $(A, B) + (B, C) = (A, C)$ look dually? Of course, we get $(a, b) + (b, c) = (a, c)$. But whereas in the point case we think of moving from A to B along the line joining them (and not crossing ∞) and next from B to C, in the dual case we must think of a line turning from a to b (and not passing O) about the common point of a and b, and next from b to c about their meeting point. The result is then the same as turning from a to c about their meeting point.
The maximal possible rotation is a half turn because the lines are not allowed to pass O.

10 Semantic considerations

The word *vector* goes back to Hamilton (1805-1865). Initially it was used to denote concepts like velocity, accelleration, force etc. Geometrically it meant a line segment together with direction, or - in modern terms - an equivalence class of the relation $(A, B) \sim (C, D)$ defined by: the line AB is parallel to line CD, $d(A, B) = d(C, D)$ and the direction from A to B is the same as that from C to D, where A, B, C and D are points in the Euclidean space or plane. Today, a vector space in mathematics is an abstract set satisfying certain axioms. Their use is widespread in mathematics.

In physics we work with distinct qualities, like time, temperature, colour etc. We have to be very careful not to confuse these. A point in physical space is a location in it. It is meaningless to add two points, or to multiply a point with a number. Full stop.

If we fix an origin O, we can convert our physical space - at least locally - into a vector space. Each point A corresponds to one vector OA. Adding two vectors, viz. $OA + OB = OC$, means: if a particle moves from O to A, and then from A to C (where the *free* vector AC equals OB), the result is the same as when it moves from O to C. But of course the two roads differ, so it is only the end results that correspond. If we take O as the location of the observer, the correspondence of physical space and vector space becomes less artificial.
In analytical geometry, location and direction vectors are very comfortable in defining point

sets like lines and planes. Yet - to our view - the essence here is coordinate geometry rather than vector geometry.

Velocities and forces are (co-) vectors by nature. It makes sense to add two velocities, or to multiply them with scalars. Now, vectors are closely related to infinitesimal calculus and differential equations. It is at the same time lucky and tragic that the *tangent space*[4] of Euclidean space is isomorphic to the space itself: lucky, because it simplifies computations a lot, tragic, because it obscures the diference in quality between points and vectors.

So, firstly, we have to distinguish between our unique physical space and all kinds of geometrical spaces. Our physical space is locally Euclidean, but *a priory* it could turn out to be any topological manifold. Let's call this space S.

Secondly, in any point P of S (like in every other differentiable manifold) is defined an abstract tangent space T_PS. This tangent space is a vector space, that is, with our new definition, a projective space in which a (hyper-) plane ∞ and a point O are singled out. Though it is comfortable to think of this tangent space T_PS as touching S in P, it is not always adequate to do so, and, indeed, at times confusing. If we consider a circle in Euclidean plane, the tangent line in a point can be found by differetiating some vector function. And if a particle moves on this circle due to a centripetal force, it will leave the circle and continue along this tangent line if - at the moment that the particle is in the tangent point - the force suddenly disappears. But the abstract tangent space consists of (velocity-) vectors, and not of points on the tangent line. That means, if the tangent space has a geometrical meaning, we can think of T_PS touching S in the point $P = O$. But if it has *not*, this image of touching spaces might obscure the true physical concepts. In that connection it may be illuminating that, according to Steiner, one enters a different - supersensual - world as soon as one changes from physical points to differentials[5].

Thirdly, though projective spaces have an immediate and historic geometric interpretation, as vector spaces in the above developed way their elements (vectors, covectors etc.) can no longer be seen as points, that is as locations in physical space. Rather one should determine their quality depending on the physical phenomena at hand: an electric force being completely different from, say, the speed of a planet.

Fourthly, it is possible that the similarity between the inproduct of two vectors (or covectors) and the action of a covector on a vector, obscures the true meaning of some physical concepts.

Finally, the cotangent space is really the same projective space T_PS, but a covector is a hyperplane in it. Now one could point out the following asymmetry. Whereas the tangent space - in geometrical situations - touches S in our above $P = O$, the dual space does not touch S in ∞. However, that is because we are used to think pointwise. If we consider S as a collection of planes, then the cotangent space $T_\alpha S$ does meet S in $\alpha = \infty$.

Yet it is an interesting question to ask: what is the meaning of ∞ if we are investigating electric forces, or velocities, or whatever physical vectors. But that is for physicists to investigate, rather than for mathematicians.

[4]See any introductory course in differential topology, e.g. J.M. Lee, Introduction to Smooth Manifolds, 2006 Springer

[5]See e.g. R. Steiner, *Mathematik und Okkultismus*, essay in GA 35