Imaginary elements in geometry

according to Felix Klein

In this part we will use uppercase letters for points as well as for planes. Lowercase ones indicate lines or variable elements (which can be points as well as lines as well as planes). Greek letters denote (real or imaginary) numbers.

1 The space over the reals

Let $\mathcal{S}$ be the 3-dimensional projective space over the real numbers. Let $\mathcal{S}_0$ be the set of its points, $\mathcal{S}_1$ the set of lines and $\mathcal{S}_2$ the set of planes. We add two extra elements to $\mathcal{S}$, viz. the empty set $\emptyset$ and the whole space $\mathbb{P} = \mathbb{P}_3$ as an entity; for each other element x holds $\emptyset \prec x \prec \mathbb{P}$. Now by definition

$$\mathcal{S} = \{\emptyset\} \cup \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{\mathbb{P}\}$$

We say that the *dimension* of $\emptyset$ is -1, that of a point 0, that of a line 1, that of a plane 2 and that of the entire space 3. An open interval $\langle a, b\rangle$ of $\mathcal{S}$ is the set of elements between a and b:

$$\langle a, b\rangle = \{x \in S | a \prec x \prec b\}$$

In particular, if P is a point, l a line and A a plane with $P \prec l \prec A$, we have
$\langle \emptyset, l\rangle$ is the collection of points on l,
$\langle \emptyset, A\rangle$ is the field of points and lines in A,
- $\langle \emptyset, A\rangle_0 = \langle \emptyset, A\rangle \cap S_0$ is the field of points in A,
- $\langle \emptyset, A\rangle_1 = \langle \emptyset, A\rangle \cap S_1$ is the field of lines in A,
$\langle \emptyset, \mathbb{P}\rangle = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$,
$\langle P, A\rangle$ is the pencil of lines in A through P,
$\langle P, \mathbb{P}\rangle$ is the bundle of lines and planes through P,
- $\langle P, \mathbb{P}\rangle_1 = \langle P, \mathbb{P}\rangle \cap S_1$ is the bundle of lines through P,
- $\langle P, \mathbb{P}\rangle_2 = \langle P, \mathbb{P}\rangle \cap S_2$ is the bundle of planes through P,
$\langle l, \mathbb{P}\rangle$ is the pencil of planes through l.

We will no longer use the name of a line for the collection of points on it, nor use the name of a point for the set of lines through it; but we'll use the proper open interval. We will abbreviate $f : \langle x, y\rangle \to \langle x, y\rangle$ by $f_{\langle x,y\rangle}$. But if P, Q, R are distinct points on a line f_{PQR} will still mean the Klein-map defined by these points, and dually if A, B, C are distinct planes through one line, f_{ABC} will mean the Klein-map defined by these planes.

2 Low imaginary elements

We are going to define imaginary elements of $\mathcal{S}$. This depends heavily on what is called 'line geometry', which is about lines in 3-space, ruled surfaces (especially the hyperboloid) and – above all – linear congruences. For more information on line geometry see for instance [Ziegler2012] or the oldies [Staudt1847] and [Reye1899].

As before, an *imaginary point* is a projective map $P = P_{\langle \emptyset, l \rangle}$ with the properties $l \in \mathcal{S}_1$, $P \neq 1_{\langle \emptyset, l \rangle}$ and $P^3 = 1_{\langle \emptyset, l \rangle}$. The collection of imaginary points is $\mathcal{T}_0$. Dually, an *imaginary plane* is a projective map $A = A_{\langle l, \mathbb{P} \rangle}$ with the properties $l \in \mathcal{S}_1$, $A \neq 1_{\langle l, \mathbb{P} \rangle}$ and $A^3 = 1_{\langle l, \mathbb{P} \rangle}$. The collection of imaginary planes is $\mathcal{T}_2$.

A *low imaginary line* (or a *line of type I*) is a projective map $l = l_{\langle P, A \rangle}$ with the properties $P \in \mathcal{S}_0$, $A \in \mathcal{S}_2$, $P \prec A$, $l \neq 1_{\langle P, A \rangle}$ and $l^3 = 1_{\langle P, A \rangle}$. The collection of low imaginary lines is $\mathcal{T}_i$. If l, l' and l'' are concurrent lines in a plane, then $f_{ll'l''}$ is the low imanigary line defined by them.

3 The high imaginary line

From 2-dimensional geometry we know that any two distinct points in the plane determine one connecting line. In particular two imaginary points in the plane, $P_{\langle \emptyset, l \rangle}$ and $Q_{\langle \emptyset, m \rangle}$ with $l \neq m$, determine one imaginary line, viz a low one. But if the two lines are in 3-space and skew, the construction of that joining line (see last part of the proof of proposition **??**) is no longer possible. This situation leads to a new type of lines, the high imaginary ones.

Recall that a *hyperboloid* $\mathcal{H}$ is a quadratic surface containing two pencils of lines, $\mathcal{H}_a$ (or $\mathcal{H}_{\text{red}}$ or $\mathcal{H}_{\text{right}}$) and $\mathcal{H}_b$ ($= \mathcal{H}_{\text{blue}} = \mathcal{H}_{\text{left}}$; Von Staudt called them *Regelschaar* and *Leitschaar*). Each line of the first pencil meets every line of the second and vice versa. Each pair of distinct lines from one pencil is skew. The join of two distinct meeting lines $l, m \in \mathcal{H}$ is a tangent plane of it. Each line of the pencil $\langle l \wedge m, l \vee m \rangle$ is a tangent of $\mathcal{H}$, l and m being very special tangents.
So, given $\mathcal{H}_a$ and $\mathcal{H}_b$, a precise definition of $\mathcal{H}$ would be $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2$ with $\mathcal{H}_1 = \mathcal{H}_a \cup \mathcal{H}_b$, $\mathcal{H}_0 = \{l \wedge m | l \in \mathcal{H}_a,\ m \in \mathcal{H}_b\}$ and $\mathcal{H}_2 = \{l \vee m | l \in \mathcal{H}_a,\ m \in \mathcal{H}_b\}$.

Let l, l', l'' be pairwise skew lines. We are going to define a projective map $f = f_{ll'l''}$ that moves l to l', l' to l'' and l'' to l, and as always $f^3 = 1$. It will appear that f can be defined in all points, lines and planes *in a natural way*. First we observe that l, l', l'' define a unique hyperboloid $\mathcal{H}$. Let $\mathcal{H}_a$ (the red lines in figure 1) be its pencil containing l, l' and l'' and let u be any line of the other pencil, $\mathcal{H}_b$ (the blue lines). Define points $P = l \wedge u$, $P' = l' \wedge u$, $P'' = l'' \wedge u$ and planes $A = l \vee u$, $A' = l' \vee u$, $A'' = l'' \vee u$. The points define a Klein-map $f_{\langle \emptyset, u \rangle}$ and the planes define another one $f_{\langle u, \mathbb{P} \rangle}$. These maps we define to be the restictions of f to the points and planes of u. So, for all lines of $\mathcal{H}_b$, we defined the restrictions of f to the respective points an planes. Observe that the lines of $\mathcal{H}_b$ themselves are invariant under f.

From the main theorem of projective geometry[1] now follows that there is a unique projective map that extends our f to the entire space. This can be seen as follows. Take any three lines $a_1, a_2, a_3 \in \mathcal{H}_a$ and any three lines $b_1, b_2, b_3 \in \mathcal{H}_b$. These lines meet in 9 distinct points $X_{ij} = a_i \wedge b_j$ of H. These points are of course not in general position, but $X_{11}, X_{12}, X_{22}, X_{23}, X_{33}$

[1] In three dimensions this is: given 5 points P_i in general position, and another five Q_i in general position, then there is exactly one projective map that maps P_i on Q_i for all i.

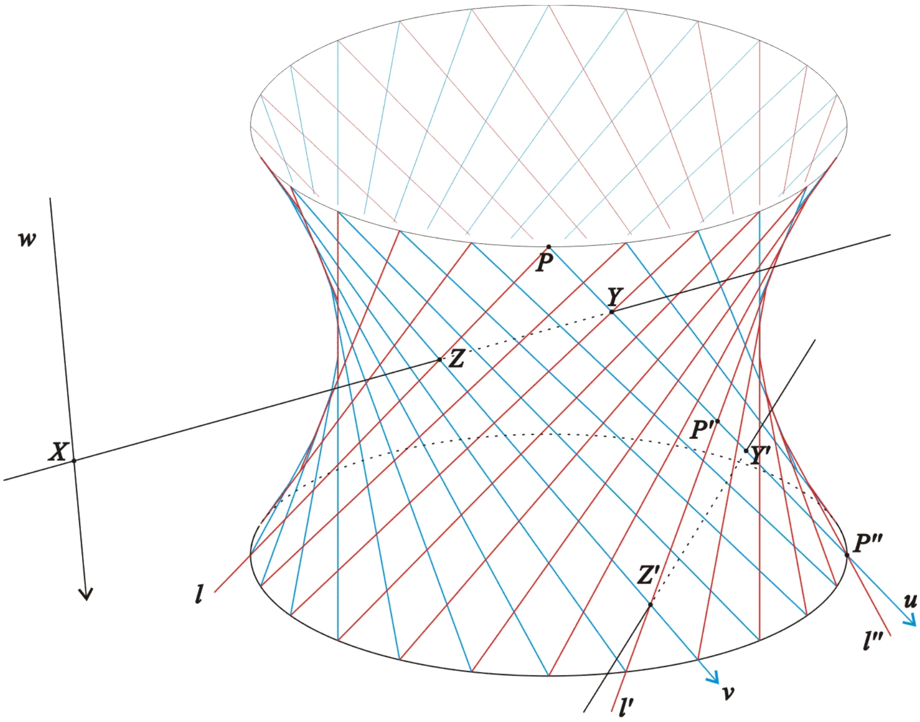

Figure 1: construction of a high imaginary line

are, see figure 2. And so are their images under f. Thus from the main theorem we know that f has a unique extension.

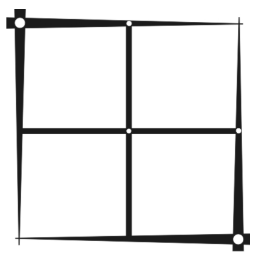

Figure 2: five points in general position

Now take any point X not on $\mathcal{H}$. We are going to construct $f(X)$. Take two distinct lines u, v from $\mathcal{H}_b$. The plane $X \vee v$ meets u in a point Y, so XY is the transversal of u, v through X. Let Z be the meeting point of XY and v. Since $u, v \in \mathcal{H}_b$, f is defined on both. Let $Y' = f(Y)$, $Y'' = f(Y')$, $Z' = f(Z)$, $Z'' = f(Z')$. The lines YZ, $Y'Z'$ and $Y''Z''$ determine a second hyperboloid $\mathcal{L}$, viz. a pencil $\mathcal{L}_a$ of lines. Let $\mathcal{L}_b$ be the corresponding second pencil and let $x \in \mathcal{L}_b$ be the line throug X. Define $X' = Y'Z' \wedge x$ and $X'' = Y''Z'' \wedge x$ and let $f_{\langle \emptyset, x \rangle} = f_{XX'X''}$. Thus we found the image under f of each point of space. The line x itself is again invariant under f.

In a similar way we can construct the image of a plane that is not in $\mathcal{H}_2$. And the image of an arbitrary line n is the join of the images of any two distinct points on n, or the meet of the images of any two distinct planes containing n.

Obviously there are no invariant real points nor planes. But the lines of $\mathcal{H}_b$ and $\mathcal{L}_b$ are invariant. And since X was an arbitrary point, each real point (dually: each real plane) is on (dually: contains) one invariant line of f. The invariant real lines form an *elliptic*[2] *linear congruence* $\mathcal{C}$. If a line m does not belong to $\mathcal{C}$, i.e. $f(m) \neq m$, then the lines of $\mathcal{C}$ that meet m form a pencil $\mathcal{L}_b$ of a hyperboloid $\mathcal{L}$, and the lines m, $f(m)$ and $f^2(m)$ belong to its pencil $\mathcal{L}_a$.

Definition 3.1 *A* high imaginary line *(or a* line of type II*) is a non-trivial projective map $f : \mathcal{S} \to \mathcal{S}$ with the following properties:*
- $f^3 = 1$
- f has no invariant real points, nor planes
- the invariant real lines of f form an elliptic linear congruence.

As a consequence we have the following properties:
- If P is an arbitrary real point, then P, $f(P)$ and $f^2(P)$ are on a line of the congruence.
- If A is an arbitrary real plane, then A, $f(A)$ and $f^2(A)$ share a line of the congruence.
- if l is an arbitrary real line, then either l belongs to the congruence, or l, $f(l)$ and $f^2(l)$ generate one pencil of lines of a hyperboloid, and the other pencil of this hyperboloid belongs to the congruence.

It is not so easy to get an image of the elliptic linear congruence. In section **??** of the appendix we provide some pictures that may help, and section **??** gives an alternative way to look at it.

The main reason that we introduced the high imaginary line was to extend the join-operator $\vee$ to a pair of imaginary points on skew lines and to extend the meet-operator $\wedge$ to imaginary planes on skew lines. That we succeded will be shown in proposition 6.2.

4 The matrix of a high imaginary line

Let again be givven three pairwise skew lines l, l', l'', and three distinct lines m, m', m''. Each of the second triple meets each of the first triple, see figure 3. Define $X_0 = l \wedge m$, $X_1 = l \wedge m''$, $X_2 = l'' \wedge m''$, $X_3 = l'' \wedge m$ and $U = l' \wedge m'$. Take these points as a system of reference in the usual way. Observe that plane $X_0X_1U[0:0:1:-1]$ meets line X_2X_3 in $T(0:0:1:1)$ and that $UT \wedge X_0X_1 = S(1:1:0:0)$. Finally we have $V = l' \wedge m = (1:0:0:1)$ and $W = l' \wedge m'' = (0:1:1:0)$. If f is again the Klein-map that moves l to l', l' to l'' and that leaves m, m' and m'' invariant, then it is easy to find that its matrix equals

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

[2] There are also parabolic and hyperbolic linear congruences.

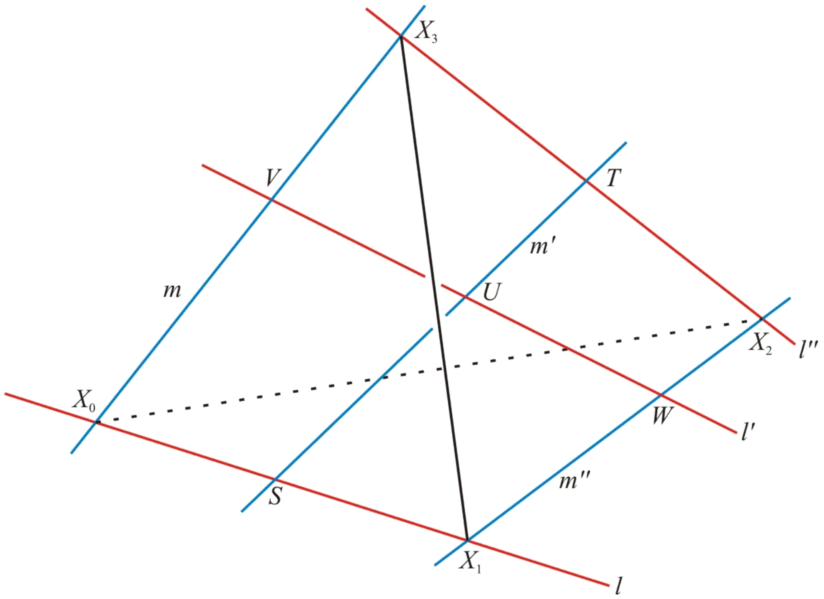

Figure 3: the matrix of a high imaginary line

The characteristic equation of f is $(\lambda^2 - \lambda + 1)^2 = 0$ and the eigenvalues are $(1 \pm i\sqrt{3})/2$. Now consider the change of basis

$$M = \frac{1}{2}\begin{pmatrix} -1+i\sqrt{3} & 0 & 0 & 2 \\ -\sqrt{3}-i & 0 & 0 & 2i \\ 0 & -1+i\sqrt{3} & 2 & 0 \\ 0 & -\sqrt{3}-i & 2i & 0 \end{pmatrix}$$

and the matrix $g =$

$$\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} & & \\ \sqrt{3} & 1 & & \\ & & 1 & -\sqrt{3} \\ & & \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & & \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & & \\ & & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ & & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$$

Verify that $Mf = gM$, hence $MfM^{-1} = g$. That is, the matrices f and g are similar, which implies – among other things – that they have the same eigenspace structure. We will investigate g in more detail in section **??**.

5 The real and imaginary parts

Let the collection of high imaginary lines be $\mathcal{T}_{ii}$, and the collection of all imaginary lines be $\mathcal{T}_1$. Hence $\mathcal{T}_1 = \mathcal{T}_i \cup \mathcal{T}_{ii}$. Let the collection of all imaginary elements be

$$\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$$

Thus a new space, the *complex 3-dimensional projective space*, is defined as the set $\mathcal{U} = \mathcal{S} \cup \mathcal{T}$. The elements of $\mathcal{S}$ are called real, those of $\mathcal{T}$ imaginary. If $f \in \mathcal{T}$ then $f^2 = f^{-1}$ is called the *conjugate* of f. Complex conjugation is also defined on $\mathcal{S}$, but equals the identity there.

The sets $\mathcal{T}_0$, $\mathcal{T}_2$, $\mathcal{T}_i$, $\mathcal{T}_{ii}$ are pairwise disjoint, and also $\mathcal{S}$ and $\mathcal{T}$ are disjoint. This implies that the equality relation (=) is well-defined in the new space.

Another drawback of our new theory is the fact that so far, imaginary elements were maps of 1-dimensional intervals: an imaginary point is a map of points on a line, an imaginary plane is a map of planes around a line, a low imaginary line is a map of lines around a point in a plane. But a high imaginary line is a map of the entire projective 3-space, not even one of a 2-dimensional interval.
This is hardly compatible with the homogeneity of projective spaces from a synthetic point of view.

How many imaginary elements are there? As we know, 3-dimensional real projective space has ∞^3 points, ∞^4 lines and ∞^3 planes. How are these numbers in the new complex space? Each real line contains ∞^2 imaginary points, so the whole space contains ∞^6 imaginary points. Together with the real points there are ∞^6 points in $\mathcal{U}$. The same numbers hold for planes.

There are ∞^3 real planes in space, each containing ∞^2 real points. That gives ∞^5 distinct pencils. Each pencil contains ∞^2 imaginary lines. So the total number of low imaginary lines is ∞^7. Thake two distinct planes in space. Each line that is not in one or both of them, meets each of these planes in a point. Each plane has ∞^4 points, so there are ∞^8 of these lines. The lines in the planes add up to $\infty^4 + \infty^4 - 1$ so the total number of lines in space is ∞^8. Since there are 'only' ∞^4 real and ∞^7 low imaginary lines, there must be ∞^8 high imaginary ones.

Space	Points	Lines I	II	Planes
3-d				
$\mathcal{S}$	∞^3	∞^4	0	∞^3
$\mathcal{T},\mathcal{U}$	∞^6	∞^7	∞^8	∞^6
2-d				
$\mathcal{S}$	∞^2	∞^2		
$\mathcal{T},\mathcal{U}$	∞^4	∞^4		
1-d				
$\mathcal{S}$	∞^1			
$\mathcal{T},\mathcal{U}$	∞^2			

6 Incidence

We are going to extend the ordering relation $\prec$ ($\succ$) to our new space. We have to make sure that in each real plane the 2-dimensional relations – as defined in **??** – hold. These are:

1 The imaginary point $P_{\langle\emptyset,l\rangle}$ lies on its real base line l. It is on no other real line in space.

2 A low imaginary line $l_{\langle P,A\rangle}$ (where necessarily point P is in plane A) is said to go through its real base point P and to lie in its base plane A. It has no other real points nor planes.

3 The imaginary point $P_{\langle\emptyset,l\rangle}$ is on the low imaginary line $m_{\langle Q,A\rangle}$ iff[3]:
- $Q \not\prec l \prec A$, **and**
- $\forall X \in \langle\emptyset, l\rangle : P(X) \vee Q = m(X \vee Q)$ **or** $\forall x \in \langle Q, A\rangle : P(x \wedge l) = m(x) \wedge l$.

In addition we want the dual statements to hold in space (the second one is selfdual):

1* The imaginary plane $A_{\langle l,\mathbb{P}\rangle}$ contains its real base line l. It contains no other real line in space.

3* The imaginary plane $A_{\langle l,\mathbb{P}\rangle}$ contains the low imaginary line $m_{\langle P,B\rangle}$ iff:
- $P \prec l \not\prec B$ **and**
- $\forall X \in \langle l, \mathbb{P}\rangle : A(X) \wedge B = m(X \wedge B)$ **or** $\forall x \in \langle P, B\rangle : m(x) \vee l = A(x \vee l)$.

And of course we want to keep the fundamental rules: two distinct points have one line that connects them, and two distinct planes have one common line, etc. That leads to the following extension of $\prec$.

Definition 6.1 *of* $\prec$

1=11* real point in real line: done

2=13* real point P in low im. line $l_{\langle Q,A\rangle}$: iff $P = Q$

3=15* real point P in high im. line $l_{\langle\emptyset,\mathbb{P}\rangle}$: never

4=4* real point in real plane: done

5=9* real point P in im. plane $A_{\langle l,\mathbb{P}\rangle}$: iff $P \prec l$

6=12* im. point $P_{\langle\emptyset,l\rangle}$ in real line m: iff $l = m$

7=14* im. point $P_{\langle\emptyset,l\rangle}$ in low im. line $m_{\langle Q,A\rangle}$
iff $Q \not\prec l \prec A$ **and**
$((\forall X \in \langle\emptyset, l\rangle :\ m(X \vee Q) = P(X) \vee Q)$ **or**
$(\forall x \in \langle Q, A\rangle :\ m(x) \wedge l = P(x \wedge l))$ $(\heartsuit)$

8=16* im. point $P_{\langle\emptyset,l\rangle}$ in high im. line $m_{\langle\emptyset,\mathbb{P}\rangle}$:
iff $\forall X \in \langle\emptyset, l\rangle :\ P(X) = m(X)$

9=5* im. point $P_{\langle\emptyset,l\rangle}$ in real plane A: iff $l \prec A$

[3]'iff' means 'if and only if'

10=10* im. point $P_{\langle\emptyset,l\rangle}$ in im. plane $A_{\langle m,\mathbb{P}\rangle}$
iff $l = m$ **or** (l, m are skew **and**
$(\forall X \in \langle\emptyset, l\rangle : P(X) \vee m = A(X \vee m))$ **or**
$(\forall Y \in \langle m, \mathbb{P}\rangle : A(Y) \wedge l = P(Y \wedge l)))$ $(\heartsuit)$
(see left part of figure 4)

11=1* real line in real plane: done

12=6* real line l in im. plane $A_{\langle m,\mathbb{P}\rangle}$: iff $l = m$

13=2* low im. line $l_{\langle P,A\rangle}$ in real plane B: iff $A = B$

14=7* low im. line $l_{\langle P,A\rangle}$ in im. plane $B_{\langle m,\mathbb{P}\rangle}$
iff $P \prec m \not\prec A$ **and**
$(\forall X \in \langle m, \mathbb{P}\rangle :\ l(X \wedge A) = B(X) \wedge A$ **or**
$\forall x \in \langle P, A\rangle : B(x \vee m) = l(x) \vee m))$ $(\heartsuit)$
(see right part of figure 4)

15=3* high im. line $l_{\langle\emptyset,\mathbb{P}\rangle}$ in real plane: never

16=8* high im. line $l_{\langle\emptyset,\mathbb{P}\rangle}$ in im. plane $A_{\langle m,\mathbb{P}\rangle}$
iff $\forall X \in \langle m, \mathbb{P}\rangle :\ l(X) = A(X)$

In addition we define of course $\emptyset \prec f \prec \mathbb{P}$ for each imaginary element f. There are no other cases of $x \prec y$. $\diamond$

In the first column the cases and their duals are listed. Observe that in cases 8 and 16 the condition implies $m(l) = l$ resp. $l(m) = m$, i.e. the line l resp. m belongs to the linear congruence. In the cases marked with $(\heartsuit)$ the last two conditions are equivalent. With these

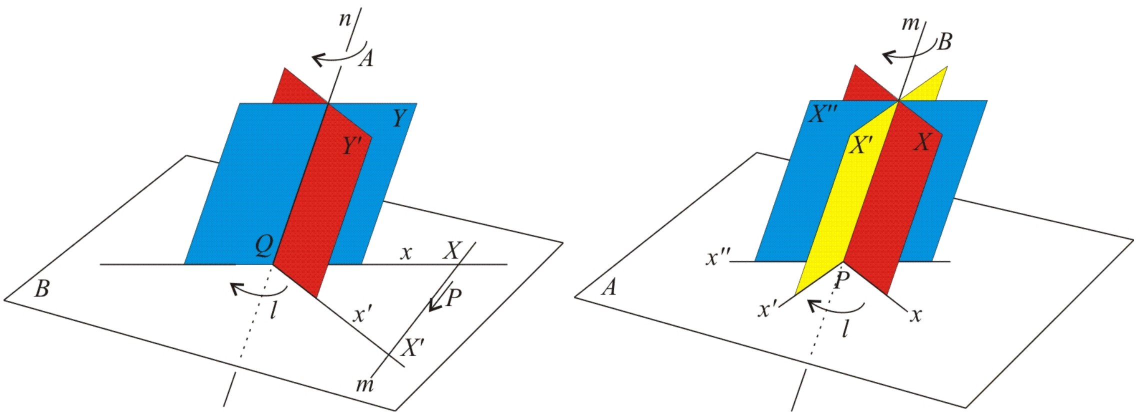

Figure 4: incidence in the Klein-space

definitions we can now prove the main *raison d'être* of the high imaginary line:

Proposition 6.2 *If m, n are skew lines, P, P', P'' distinct points on m and Q, Q', Q'' distinct points on n, there is a unique high-imaginary line that joins the points $S_{PP'P''}$ and $T_{QQ'Q''}$. Dually: if m, n are skew lines, A, A', A'' distinct planes through m and B, B', B'' distinct planes through n, there is a unique high-imaginary line that is contained in both $C_{AA'A''}$ and $D_{BB'B''}$.*

Proof. Define $l = PQ$, $l' = P'Q'$ and $l'' = P''Q''$. Suppose l, l' are in a real plane. Then also P, Q, P' and Q'. But then m and n are in that plane, against hypothesis. In the same way we see that l, l'' are skew and likewise l', l''. Then, by the construction of the linear congruence, there is a unique high imaginary line h that extends S and T. Hence, by part 8 of definition 6.1: $S \prec h$ and $T \prec h$. $\diamond$

7 The ordering properties

Observe that in the definition of $x \prec y$ the dimension of x is always smaller than that of y. So, $x \prec y$ implies $y \not\prec x$.

If we define '$x \preceq y$' as '$x \prec y$ or $x = y$', we immediately have:
- $x \preceq x$ for all x (reflexivity) and
- $x \preceq y$ and $y \preceq x$ implies $x = y$ (anti symmetry).

Proposition 7.1 *The relation $\prec$ (and hence $\preceq$) is transitive, i.e. from $x \prec y$ and $y \prec z$ follows $x \prec z$.*

Proof. We only have to prove this for the case $P \prec l$, $l \prec A$ with P a point, l a line and A a plane.

1. All real: done.

2. P, l real and $A_{\langle m,\mathbb{P}\rangle}$ imaginary. From $l \prec A$ follows with part 12 of definition 6.1 that $l = m$, hence $P \prec m$ hence – with part 5 – $P \prec A$.

3. P, A real and $l_{\langle Q,B\rangle}$ imaginary. Because l is low imaginary we have $Q \prec B$. From $P \prec l$ follows with part 2 that $P = Q$ and from $l \prec A$ with part 13 that $A = B$. Then $P = Q \prec B = A$.

4. P, A real and l high imaginary is an impossible case.

5. l, A real and $P_{\langle\emptyset,m\rangle}$ imaginary. Because $P \prec l$ we have with part 6 that $l = m$ and hence with part 9 that $P \prec A$.

6. P real, $l_{\langle Q,B\rangle}$ and $A_{\langle m,\mathbb{P}\rangle}$ imaginary. From $P \prec l$ follows with part 2 that $P = Q$ and from $l \prec A$ with part 14 that $Q \prec m$. Hence $P \prec m$ and – by part 5 – $P \prec A$.

7. P real, l high and $A_{\langle m,\mathbb{P}\rangle}$ low imaginary is again impossible.

8. l real, $P_{\langle\emptyset,m\rangle}$ and $A_{\langle n,\mathbb{P}\rangle}$ imaginary. From $P \prec l$ follows with part 6 that $l = m$ and from $l \prec A$ with part 12 that $n = l$. Hence $m = n$ and from part 10 we infere that $P \prec A$.

9. A real, $P_{\langle\emptyset,m\rangle}$ and $l_{\langle Q,B\rangle}$ imaginary. From $p \prec l$ follows with part 7 that $m \prec B$ and from $l \prec A$ with 13 that $A = B$. Hence $m \prec A$. Then from part 9 follows $p \prec A$.

10. A real, $P_{\langle\emptyset,m\rangle}$ imaginary and l high imaginary is impossible.

11. $P_{\langle\emptyset,m\rangle}, l_{\langle Q,B\rangle}, A_{\langle n,\mathbb{P}\rangle}$ imaginary. From $P \prec l$ follows with part 7 that $Q \not\prec m \prec B$ and $P(X)\vee Q = l(X\vee Q)$ for all X on m. From $l \prec A$ follows with part 14 that $Q \prec n \not\prec B$ and $A(y\vee n) = l(y)\vee n$ for all $y \in \langle Q, B\rangle$. Then necessarily l, m are skew. Let X be an arbitrary point of m and let $y = X \vee Q$. Then $P(X) \vee n = P(X) \vee Q \vee n = l(X \vee Q) \vee n = l(y) \vee n = A(y \vee n) = A(X \vee Q \vee n) = A(X \vee n)$. Hence with part 10: $P \prec A$. See also left part of figure 4.

12. $P_{\langle\emptyset,m\rangle}$, $A_{\langle n,\mathbb{P}\rangle}$ low and l high imaginary. From $P \prec l$ follows with part 8 that $P(X) = l(X)$ for all X on m, and also that $l(m) = m$. From $l \prec A$ follows with part 16 that $A(Z) = l(Z)$ for all Z containing n, and also that $l(n) = n$. Since l is a projective map, it respects the join: $l(x \vee y) = l(x) \vee l(y)$. Now let X be an arbitrary point of m and let $Z = X \vee n$. Then $A(X \vee n) = A(Z) = l(Z) = l(X \vee n) = l(X) \vee l(n) = l(X) \vee n = P(X) \vee n$. From this follows with part 10 that $P \prec A$. $\diamond$

8 Check of the axioms

We will check if the space defined so far, satisfies the axioms of projective geometry. We will do this using the axioms of [Boer2009]:

Definition 8.1 *A* projective space *is a quadruple* $(\mathcal{S}, n, \dim, \preceq)$ *in which* $\mathcal{S}$ *is a set,* $n \geq 3$ *is an integer,* $\dim : \mathcal{S} \to \{-1, 0 \ldots n\}$ *is a surjective function, and* $\preceq$ *is a binary relation on* $\mathcal{S}$*, satisfying*

1 the axiom of order:
- $x \preceq x$
- $(x \preceq y$ *and* $y \preceq x) \Rightarrow x = y$
- $(x \preceq y$ *and* $y \preceq z) \Rightarrow x \preceq z$

2 the axiom of monotone dimension:
for every $x, y \in \mathcal{S}$ *we have* $x \preceq y \Rightarrow \dim(x) \leq \dim(y)$

3 the axiom of border:
- *there is an element* $\mathbf{0}$ *such that for every* $x \in \mathcal{S} : \mathbf{0} \preceq x$
- *there is an element* $\mathbf{1}$ *such that for every* $x \in \mathcal{S} : \mathbf{1} \succeq x$

4 the Lattice axiom:
- *each pair of elements of* $\mathcal{S}$ *has a least upper bound (lub, join,* $\vee$*)*
- *each pair of elements of* $\mathcal{S}$ *has a greatest lower bound (glb, meet,* $\wedge$*)*

5 the axiom of sufficient points/hyperplanes:
- *for every pair* a, b *of elements of* $\mathcal{S}$ *for which* $a \prec b$*, there exists a point*[4] x *such that* $x \not\preceq a$ *and* $x \preceq b$
- *for every pair* a, b *of elements of* $\mathcal{S}$ *for which* $a \prec b$*, there exists a hyperplane* y *such that* $a \preceq y$ *and* $b \not\preceq y$

[4] A point is an element of dimension 0, a line has dimension 1, a dual line has dimension $n-2$ and a hyperplane $n-1$.

6 the axiom of composition:
- if x is a point and a any element not containing x, then

$$\dim(a \vee x) = \dim(a) + 1$$

- if y is a hyperplane and b any element not in y, then

$$\dim(b \wedge y) = \dim(b) - 1$$

7 the axiom of cardinality:
- every line has at least three points on it
- every dual line is contained in at least three hyperplanes

In addition, 1- and 2-dimensional closed intervals of higher dimensional projective spaces, are called projective spaces as well.

With these axioms we can derive by synthetic means a vector space V over some division ring (skew field) F and it appears that each projective space is isomorphic to $V^{n+1}/\sim$ for some positive $n \in \mathbb{Z}$. Then an extra axiom states which ring is taken[5]. In our case it will be $\mathbb{C}$.

8.1 The first three axioms

Checking these axioms, we first admit to have defined a space $\mathcal{S} \cup \mathcal{T}$ and a dimension function dim from this space onto the set $\{-1, 0, 1, 2, 3\}$. Built in is the priciple of duality, so we only have to chekc one half of each doubly stated axiom. We also defined $\prec$ and $\preceq$. The last relation is reflexive, anti-symmetric and – by proposition 7.1 – transitive.

We already included in our space $\emptyset$ as smallest and $\mathbb{P}$ as biggest element. So the axiom of border is fulfilled. A scan of definition 6.1 shows that the axiom of monotone dimension is fulfilled as well.

8.2 The lattice axiom

Now we have to check that each pair of elements has a unique join. For real elements that is already guaranteed, and for two coincident elements it is trivially true. Trivially true it is also if one of the elements is the empty set or the entire space. The commutativity of $\vee$ once more reduces the number of cases. In the next table P stands for point, l for line, A for plane, i for imaginary, ii for high imaginary; s means true by symmetry, r because of realness, t

[5]See [Artin1957] or [Boer2009]

because it is trivial.

$\vee$	$\emptyset$	P	Pi	l	li	lii	A	Ai	$\mathbb{P}$
$\emptyset$	r	r	t	r	t	t	r	t	r
P	s	r	1	r	2	3	r	4	r
Pi	s	s	5	6	7	8	9	10	t
l	s	s	s	r	11	12	r	13	r
li	s	s	s	s	14	15	16	17	t
lii	s	s	s	s	s	18	19	20	t
A	s	s	s	r	s	s	r	21	r
Ai	s	s	s	s	s	s	s	22	t
$\mathbb{P}$	s	s	s	s	s	s	s	s	r

The 22 numbered boxes are yet to be proven, which is a tedious and boring job. However, since $x \vee x = x$ we can restrict to distinct elements.

1 Let P be a real point and $Q_{\langle \emptyset, l\rangle}$ imaginary, with l a real line. This is plane geometry and part of proposition **??** on page **??**.
(a) If $P \prec l$ the join is l.
(b) Else it is the low imaginary line $g_{\langle P, P\vee l\rangle}$ that is perspective with Q.

2 Let P be a real point and $l_{\langle Q,A\rangle}$ a low imaginary line. There are three cases.
(a) If $P = Q$ then $P \prec l$ and trivially the join is l,
(b) If $P \neq Q$ and $P \prec A$ then it is A, for A is an upperbound and no line nor any other plane can be an upper bound.
(c) $P \not\prec A$. The join is the imaginary plane with axis PQ that is perspective with l.

3 Let P be a real point and l a high imaginary line. Since $P \not\prec l$ the join must have at least dimension 2. Let c be the unique line of the congruence of l through P. Then P is on the imaginary plane $A = l|_c$, which then is the join: no real plane contains l, no other imaginary plane contains both l and P.

- **4, 9, 10** Let P be a point and A a plane. Then either $P \prec A$, in which case the join is A, or $P \not\prec A$ and then the join is $\mathbb{P}$.

5 Let be given the distinct imaginary points $P_{\langle \emptyset, l\rangle}$ and $Q_{\langle \emptyset, m\rangle}$.
(a) If $l = m$ then this line is the join.
(b) If l, m are in a real plane A, the join is A; this is plane geometry and part of proposition **??** again.
(c) If l, m are skew the join is a high imaginary line by proposition 6.2.

6 Let $P_{\langle \emptyset, l\rangle}$ be an imaginary point and m a real line.
(a) $l = m$. $P \prec m$ hence m is the join.
(b) $l \wedge m$ is a point S and $l \vee m$ is a plane A. A is an upper bound an no line and no other plane can be one. Hence A is the join.
(c) l, m are skew. Let $A_{\langle m, \mathbb{P}\rangle}$ be the imaginary plane that is perspective with P. Then A is an upper bound and no other plane nor any line can be one. Hence A is the join.

7 Let $P_{\langle\emptyset,l\rangle}$ be an imaginary point and $m_{\langle Q,A\rangle}$ a low imaginary line.
(a) l meets A in a point $R \neq Q$. Let $a = QR$, $a' = m(a)$, $a'' = m(a')$, $R' = l(R)$ and $R'' = l(R')$. Define $B' = a' \vee R'$, $B'' = a'' \vee R''$, $n = B' \wedge B''$, $B = n \vee a$. Then $Q \prec n$ and $D_{BB'B''}$ is the join.
(b) l meets A in a Q. Now the imaginary plane with axis l and perspective with m is the join.
(c) $l \prec A$ and $Q \not\prec l$ and P and m are not perspective. Then A is the join.
(d) $l \prec A$, $Q \prec l$. Then A is the join.
(e) $l \prec A$ and $Q \not\prec l$ and P and m are not perspective. Then $P \prec m$ and m is the join.

8 Let be given the imaginary point $P_{\langle\emptyset,l\rangle}$ and the high imaginary line $m_{\langle\emptyset,\mathbb{P}\rangle}$.
(a) If $P \prec m$ then l belongs to the linear congruence C of m and $A = m|_{\langle l,\mathbb{P}\rangle}$ is the join.
(b) If $P \not\prec m$ we take an arbitrary point Q on l and let $Q' = P(Q)$ and $Q'' = P(Q')$, so $P = P_{QQ'Q''}$, see upper part of figure 5. Let $c_0 \in C$ be the line through Q. Take any line n not in the plane $l \vee c_0$ and let $B = l \vee n$. Take a point R on n close to Q. Let c be the line of C through R. Consider the the planes $A = c \vee Q$, $A' = m(A_R)$ and $A'' = m(A')$. These planes meet B in the red lines of the figure. If we move R away from Q on n, we will reach a position R' and a congruence line c' so that one of A' and A'', say A' will reach Q' (middel part of the figure). Next we turn n in B around Q, repeat the above and find a R'_n for each n. These points R'_n form a conic and the locus of the meeting point of A'' and l is l itself (lower part of the figure). That means, there is one line n such that $A'' \wedge l = Q''$. For this position of n and c_R the imaginary plane $D_{AA'A''}$ is perspective with P, hence $P \prec A$. By construction also $m \prec D$. If on the contrary A'' contains Q' and $A' \succ Q''$, we have to take the initial R at the other side of Q, and move it away from Q.

11 Let l be a real, and $m_{P,A}$ a low imaginary line.
(a) l meets A in a point $Q \neq P$. The dimension of the join is at least 2. There is neither a real nor an imaginary plane containing them both. So the join is $\mathbb{P}$ and the lines are skew.
(b) l meets A in P. The imaginary line with axis l that is perspective with m is the join.
(c) $l \prec A$ but $P \not\prec l$. The join is A.
(d) $l \prec A$ and $P \prec l$. The join is A.

12 Let l be a real, and $m_{P,A}$ a high imaginary line.
(a) l is a line of the congruence. Then the join is the imaginary plane with axis l that is the restriction to l of m.
(b) l is not a line of the congruence. There is no real plane containing m, and an imaginary plane containing l has l as axis, hence does not contain m. So the lines are skew and the join is $\mathbb{P}$.

- **13, 16, 17, 19, 20** If l is a line and A a plane then either $l \prec A$, in which case the join is A, or else the join is $\mathbb{P}$.

14 Let $l_{\langle P,A\rangle}$ and $m_{\langle Q,B\rangle}$ be distinct low imaginary lines.
(a) $A \neq B$ and P nor Q are on the meeting line $n = A \wedge B$; the lines are not perspective.

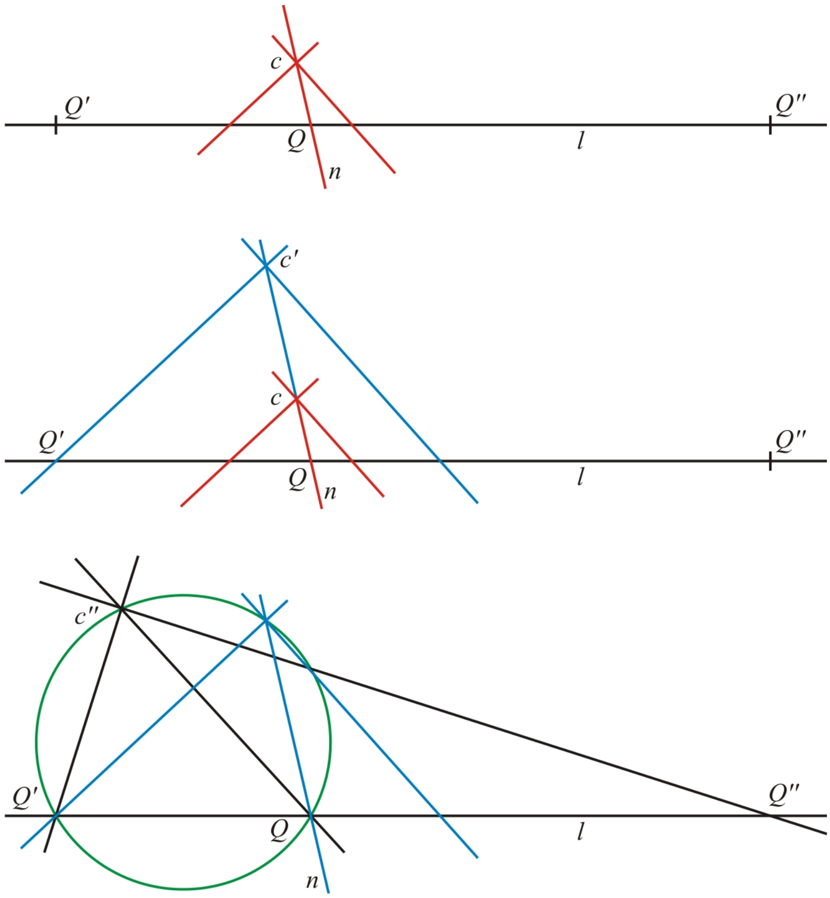

Figure 5: the join of an imaginary point and a high imaginary line

Then the lines are skew and the join is $\mathbb{P}$.
(b) The same as before, but now the lines ar perspective. The imaginary plane with axis n and perspective with both l and m is the join.
(c) Again $A \neq B$ but either $P \prec n, Q \not\prec n$ or $P \not\prec n, Q \prec n$ or $P \prec n, Q \prec n, P \neq Q$. The lines are skew and the join is $\mathbb{P}$.
(d) Again $A \neq B$ but now $P = Q \prec n$. Define $n'_A = l(n)$, $n'_B = m(n)$, $n''_A = l(n'_A)$, $n''_B = m(n'_B)$, $D' = n'_A \vee n'_B$, $D'' = n''_A \vee n''_B$, $p = D' \wedge D''$ and $D = p \vee n$. Then the imaginary plane $F_{DD'D''}$ is the join of l and m.
(e) If $A = B$ then the join is A, also if $P = Q$ (but $l \neq m$ by hypothesis.

15 let $l_{\langle P,A \rangle}$ be a low imaginary line and m a high imaginary one. Their join is at least a plane, viz. an imaginary one since no real plane contains m. Then necessarily the axis of this plane contains P but cannot lie in A, and l must be perspective with m. In all other cases the lines are skew and the join is $\mathbb{P}$.

18 Let l and m be distinct high imaginary lines. No real plane contains any high imaginary line. If A is an imaginary plane, its axis must be a line of both congruences. So if l and m share exactly one congurence line n and if $l|_n = m|_n$ then this plane A is the join. In

all other cases the lines are skew and the join is $\mathbb{P}$.

- **21, 22** Let A, B be planes. Then by hypothesis $A \neq B$ and the join is $\mathbb{P}$.

By duality we now know that each pair of elements also has a unique meet or glb.

8.3 The remaining axioms

Less work is checking the next axiom, 9, of sufficient points/hyperplanes: if $a \prec b$, there is a point X such that $X \not\prec a$ and $X \prec b$. Necessarily $\dim a < \dim b$. The cases that $a = \emptyset$ or $b = \mathbb{P}$ are trivial. The cases that both a and b are real can be skipped. And if the axiom holds for $a \prec b$ and for $b \prec c$, it certainly holds for $a \prec c$, so we can restrict to cases that $\dim b = \dim a + 1$.
The remaining cases are (1) a real point in a low imaginary line, (2,3,4) an imaginary point in a real, low imaginary or high imaginary line, (5) a real line in an imaginary plane, (6,7) a low imaginary line in a real or imaginary plane, (8) a high imaginary line in an imaginary plane and an imaginary plane in $\mathbb{P}$.
We prove nr. 4. Let $P_{\langle\emptyset,l\rangle}$ be an imaginary point in a high imaginary line m. Then l is a line of the congruence of m. Take any other line n of this congruence and take any imaginary point Q of m on n. Then Q satisfies the axiom.
The other cases are left for the reader. By duality we have the other part of the axiom: if $a \succ b$, there is a plane Y such that $Y \not\succ a$ and $Y \succ b$.

Also as an exercise for the reader we leave to check the axiom of composition (12 cases), which tells: if a is any element and X a point not in a then $\dim a \vee X = \dim a + 1$, and dually: if a is any element and ξ a hyperplane not containing a then $\dim a \wedge X = \dim a - 1$. The first part can be proven by checking the relevant cases in section 8.2.

Since we started with real geometry, each real line has already infinite many real points on it, and each imaginary line has ∞^2 points on it; so the axiom of cardinality is fulfilled.

From the axioms now follows that our space is isomorphic to a vector space over some skew field. This field is likely to contain the real numbers as a proper subset. In chapter **??** we will establish an isomorphism between the Klein-space and the numerical complex space. From that we infer that the field is $\mathbb{C}$.

References

[Artin1957] Emile Artin: *Geometric Algebra*; Interscience, New York 1957, 1961

[Boer2009] L.A.D. de Boer: *On the fundamentals of Geometry*; MathArt publishing, available on www.mathart.nl, Rotterdam 2009

[Reye1899] Theodor Reye: *Die Geometrie der Lage*, three volumes, Baumgärtners Buchhandlung, Leipzig 1892-1907

[Staudt1847] Georg Karl Christian von Staudt: *Die Geometrie der Lage*; Verlag der Fr. Korn'schen Buchhandlung, Nürnberg 1847

[Ziegler2012] Renatus Ziegler: *Projective Geometry and Line Geometry*; Dornach 2012

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