Projective Geometry with Projective Algebra. Transition to Clifford Double Algebras and to Metric Geometries. Space and Counterspace

Oliver Conradt

June 12, 2026

Abstract. This article introduces projective algebra Λ_n, in order to provide a complete system of axioms for projective geometry $\mathcal{P}_n$; it makes the transition from projective algebra Λ_n to Clifford double algebra Γ_n, in order to describe the metric Cayley-Klein geometries and it introduces the concepts of space and counterspace.

Mathematics Subject Classification (2010). Primary 14-02; Secondary 51A05, 51F99.

Keywords. exterior algebra, Clifford algebra, double algebra, complementary graded vector space, binary indices, projective geometry, Cayley-Klein geometries, space and counterspace.

1. Introduction

The purpose of this article is threefold. Firstly, to establish the 2^n-dimensional projective algebra Λ_n as a tool to describe projective geometry in such a way that projective duality and thus, the concept of space and counterspace is available in every detail of the treatment. Projective algebra Λ_n is a unity free exterior double algebra carrying the imprint of a graded algebra twice. As it is the case in all projective geometry, the projective algebra Λ_n is metric-free as well.

Secondly, the transition from the unity free exterior double algebras Λ_n to non-degenerate *or* degenerate Clifford double algebras Γ_n is established. By singling out a non-degenerate or degenerate quadric in the context of a

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June 12, 2026: manuscript submitted for publication

unity free exterior double algebra Λ_n, a non-degenerate or degenerate Clifford double algebra Γ_n respectively is defined. Clifford double algebras Γ_n carry the imprint of a Clifford algebra twice, i. e., there are always two Clifford products present. The transition from the metric free exterior double algebras to the Clifford double algebras reflects the Cayley-Klein construction of metric geometries. The metric version of the concept of space and counterspace is available through the Clifford double algebras Γ_n into every detail.

The third purpose of this article is to provide a mathematical language, which reflects the projective principle of duality and, thus, the concept of space and counterspace into every detail. You will find a few, rather simple applications in this article. Just the main task of it is to provide the above mentioned mathematical language. Please do not expect from this article more than a clear presentation of the mathematical language of space and counterspace.

The completely dual approach to geometric Clifford algebras was introduced in [Con00a]. It turns non-degenerate Clifford algebras with the dual Clifford product $*$ into Clifford *double* algebras. These non-degenerate Clifford double algebras can be used to represent projective geometry with an emphasis on its principle of duality. The concept of space and counterspace, closely linked to the projective principle of duality, was described in terms of the two Clifford products and in terms of the completely dual approach in [Con00b, Con08]. Applications of the concept of space and counterspace, especially to classical mechanics, can be found in [Con00b, Con08] as well.

The above mentioned articles and the book made a start in expressing the concept of space and counterspace in terms of non-degenerate Clifford *double* algebras. The downside of this beginning was, on the one hand, the limitation of the completely dual approach to non-degenerate Clifford algebras — a second Clifford product was not available in degenerate Clifford algebras — and, on the other hand, the use of metric Clifford algebras to express non-metric concepts from projective geometry. In this article both limitations are resolved. It is shown how the concept of space and counterspace is represented analytically in its projective version by unity free exterior double algebras (Definition 5.6) and in its metric version by generic Clifford double algebras (Definition 8.22).

It was RUDOLF STEINER (1861-1925), who first coined the concept of counterspace in a lecture in 1921[1] and, already before, [Ste21, Ste94] had sketched this idea by referring, among other things, to projective geometry. Subsequently, different scientists developed the concept of space and counterspace in terms of synthetic projective and synthetic metric geometries.[2] Among them were GEORGE ADAMS (1894-1963) [Ada81, Ada65] and LOUIS LOCHER-ERNST (1906-1962) [Loc80b, Loc80a, LE70], who today are considered to have independently discovered the mathematical formulation of the

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[1] [Ste97, lecture from 15th January 1921]

[2] [Con08, pp. 55-58]

concept of space and counterspace. In the second half, with an emphasis towards the end of the 20th and in the 21st century, different authors started to describe this concept analytically. [UvV59, Gsc91a, Tho08, Con08, Gun11b] We will show in this article, how the concept of space and counterspace is represented in terms of double algebras.

Section 2 introduces basic concepts such as complementary graded vector spaces as well as complementary graded algebras and defines the plus minus notation, which is used throughout the whole article. The plus minus notation was inspired by books of Louis Locher-Ernst [LE70] and Hanns-Jörg Stoss [Sto99] and may indicate, in which context a multi vector or with respect to which of the two exterior products and — later on in Section 8 — with respect to which of the two Clifford products an element of the double algebra is to be seen. The plus minus notation was already used in [Con00a, Con00b, Con08].

Section 3 shortly introduces binary numbers and some binary operators. We adopt the concept of binary indices to label elements of vector spaces and algebras from Dennis W. Marks [Mar04] and Pertti Lounesto [Lou01] throughout the whole article.

In Section 4 we introduce the unity free exterior double algebras Λ_n, which will be used to represent projective geometry $\mathcal{P}_n$. This is the reason, why these unity free exterior double algebras will be called, from Section 5 on, *projective algebras* Λ_n. — Why do these algebras need to be unity free? Two basic elements A and B, like e. g. a point A and a plane B of projective space, coincide, if and only if the two exterior products vanish,

$$A \wedge B = 0 \qquad \text{and} \qquad A \vee B = 0. \tag{1.1}$$

The lowest grade in the plus approach, which in a usual Graßmann algebra is represented by the one-element, stands in spacial geometry for the projective space of all planes as such. And the highest grade in the plus approach of the exterior algebra represents the projective space of all points as such.[3] And similar in the geometries of the planar field and the centric bundle or in the geometries of a line or a pencil of lines. See also Table 4. Since any basic element belongs to the entire space as such in either of the two possible manifestations and, thus, is incident with the latter without any exception, the lowest and the highest grade must always be divisors of zero for any basic element of the algebra. If not, the incidence relations of equation (1.1) would not be universally valid in all projective geometry $\mathcal{P}_n$.

The multiplication tables for the two exterior products $\wedge$ and $\vee$ with respect to the elements of the system of bases $\{P_{\mathbf{b}}\}$ in the plus approach to Λ_n and with respect to the elements of the system of bases $\{E_{\mathbf{b}}\}$ in the minus approach to Λ_n respectively are provided for any dimension n in terms of the coefficients $\alpha_{\mathbf{bc}}$. These coefficients take on the values ± 1 depending on the binary indices $\mathbf{b}$ and $\mathbf{c}$ of the involved basis elements. We will use them again and again in this article. Compare Theorem 4.10.

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[3] In the minus approach, the lowest grade represents the projective space of all points as such and the highest grade represents the projective space of all planes as such.

Section 4 then continues with the Definition of the four different types of algebra homomorphisms π, ρ, $\hat\pi$ and $\hat\rho$,

$$\pi(A \wedge B) = \lambda \cdot \pi(A) \wedge \pi(B), \qquad \lambda \in \mathbb{F} \setminus \{0\}, \quad \forall\, A, B \in \Lambda_n, \tag{1.2}$$

$$\rho(A \vee B) = \mu \cdot \rho(A) \vee \rho(B), \qquad \mu \in \mathbb{F} \setminus \{0\}, \quad \forall\, A, B \in \Lambda_n, \tag{1.3}$$

$$\hat\pi(A \wedge B) = \lambda \cdot \hat\pi(A) \vee \hat\pi(B), \qquad \lambda \in \mathbb{F} \setminus \{0\}, \quad \forall\, A, B \in \Lambda_n, \tag{1.4}$$

$$\hat\rho(A \vee B) = \mu \cdot \hat\rho(A) \wedge \hat\rho(B), \qquad \mu \in \mathbb{F} \setminus \{0\}, \quad \forall\, A, B \in \Lambda_n. \tag{1.5}$$

The fourfold appearance of the same kind of linear function is due to the two involved exterior double algebras Λ_n and Λ'_n: π represents a linear function from the plus approach of Λ_n to the plus approach of Λ'_n, ρ represents a linear function from the minus approach of Λ_n to the minus approach of Λ'_n, $\hat\pi$ represents a linear function from the plus approach of Λ_n to the minus approach of Λ'_n and $\hat\rho$ represents a linear function from the minus approach of Λ_n to the plus approach of Λ'_n. Later on in Section 6, the algebra isomorphisms will describe the different types of projective transformations.

In the literature on geometric Clifford algebras, this type of linear function usually is referred to as an outermorphism.[4] And it is defined with the coefficients λ and μ being 1. In Section 7 and in the context of harmonic orthonormal systems of bases, we are going to describe the non-null polarities with algebra isomorphisms and the respective quadrics in the plus and — simultaneously — in the minus approach. In order to do so, the coefficients λ and μ take on the values of 1 and -1. It is not possible to describe the non-null polarities and the respective quadrics at the same time in the plus as well as in the minus approach without the coefficients λ and μ. While it is possible to describe all non-null polarities and respective quadrics in just one approach with outermorphisms, where the coefficients λ and μ are set to 1.

Finally, for the transition from projective to Clifford double algebras in Section 8 these coefficients are essential too,

$$\mathbf{1}^+ = \mathbf{Z}^+, \qquad \frac{1}{2}\left(A^+_{\overline{1}} B^+_{\overline{1}} - B^+_{\overline{1}} A^+_{\overline{1}}\right) = A^+_{\overline{1}} \wedge B^+_{\overline{1}}, \tag{1.6}$$

$$\mathbf{1}^- = \frac{1}{\lambda}\mathbf{Z}^-, \qquad \frac{1}{2}\left(A^-_{\overline{1}} * B^-_{\overline{1}} - B^-_{\overline{1}} * A^-_{\overline{1}}\right) = \frac{1}{\mu} A^-_{\overline{1}} \vee B^-_{\overline{1}}. \tag{1.7}$$

In equation (1.6) juxtaposition denotes one of the two Clifford products. The other Clifford product is denoted by the sign $*$ in equation (1.7). Compare also equations (8.34), (8.35) and (8.38).

An outermorphism f usually transforms an homogeneous multi vector of grade 0 $A_{\overline{0}}$ by the identity matrix, i. e. $f(A_{\overline{0}}) = A_{\overline{0}}$. See e. g. [LD09, p. 101]. However, Definition and Theorem 4.33 in this article introduces the transformations of the homogeneous multi vectors of grade 0 by

$$\pi(\mathbf{Z}^+) = \frac{1 - \delta_{0\,\det\pi}}{\lambda}(\mathbf{Z}')^+, \qquad \rho(\mathbf{Z}^-) = \frac{1 - \delta_{0\,\det\rho}}{\mu}(\mathbf{Z}')^-, \tag{1.8}$$

$$\hat\pi(\mathbf{Z}^+) = \frac{1 - \delta_{0\,\det\hat\pi}}{\lambda}(\mathbf{Z}')^-, \qquad \hat\rho(\mathbf{Z}^-) = \frac{1 - \delta_{0\,\det\hat\rho}}{\mu}(\mathbf{Z}')^+. \tag{1.9}$$

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[4] Compare [LS16, Definition 3.2] and [LD09, p. 101].

$\mathbf{Z}$ and $\mathbf{Z}'$ represent the bases elements of grade 0 in the plus and in the minus approach respectively. If the algebra homomorphisms π, ρ, $\hat{\pi}$ or $\hat{\rho}$ have a vanishing determinant, then these algebra homomorphisms need to vanish for all multi vectors of grade 0. Compare Theorem 4.34 of this article. If not, there is the non-zero factor of $1/\lambda$ or $1/\mu$, which becomes 1 whenever $\lambda = 1$ or $\mu = 1$ respectively.

In projective geometry, the 2^n different homogeneous multi vectors of the basis in the plus approach $\{P_{\mathbf{b}}\}$ and the 2^n different homogeneous multi vectors of the basis in the minus approach $\{E_{\mathbf{b}}\}$ represent each a simplex of n points (hyperpoints) and n planes (hyperplanes). As soon as these two simplices are related to each other by an, in general, generic pair of simplices transformations ι and κ satisfying[5] $\iota\kappa^T = \kappa^T\iota = \mathrm{Id}$, the minor exterior product $\vee$ can also be computed in the plus approach to Λ_n and the major exterior product $\wedge$ also in the minus approach to Λ_n. Each pair of simplices transformations (ι, κ) establishes a certain *model* of the exterior double algebra Λ_n by fixing the positions of the two bases simplices with respect to each other.

We will work in this article in the *harmonic* model of the unity free exterior double algebra, where the two bases simplices become one simplex in such a way, that opposite points (hyperpoints) $P_{\mathbf{b}}$ and planes (hyperplanes) $E_{\mathbf{b}}$ carry the same index $\mathbf{b}$. The respective pair of simplices transformations (ι_0, κ_0) is given by

$$\iota_0(P_{\mathbf{b}}) = \langle E_{\mathbf{b}}\rangle^+ = \alpha_{\mathbf{b}\overline{\mathbf{b}}}P_{\overline{\mathbf{b}}}, \qquad \kappa_0(E_{\mathbf{b}}) = \langle P_{\mathbf{b}}\rangle^- = \alpha_{\overline{\mathbf{b}}\mathbf{b}}E_{\overline{\mathbf{b}}}. \tag{1.10}$$

We call these systems of bases *harmonic*, too. Compare Definition 4.26.

As a consequence of the harmonic model of Λ_n, the incidence relations of equation (1.1) are represented on the level of coordinates by the standard inner product. Compare Theorem 4.27.

More details on the geometric background of the harmonic model can be found in [Sto09]. Also the term ‘harmonic’ has been taken from that book.

Another important topic of Section 4 is summarised in Theorem 4.38. Each of the four possible types of algebra isomorphisms preserve both exterior products,

$$\pi(X \wedge Y) = \lambda \cdot \pi(X) \wedge \pi(Y), \tag{1.11}$$

$$\pi(X \vee Y) = \frac{1}{\lambda^{n-1}\det\pi} \cdot \pi(X) \vee \pi(Y), \tag{1.12}$$

$$\rho(X \vee Y) = \mu \cdot \rho(X) \vee \rho(Y), \tag{1.13}$$

$$\rho(X \wedge Y) = \frac{1}{\mu^{n-1}\det\rho} \cdot \rho(X) \wedge \rho(Y), \tag{1.14}$$

$$\hat{\pi}(X \wedge Y) = \lambda \cdot \hat{\pi}(X) \vee \hat{\pi}(Y), \tag{1.15}$$

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[5] Compare Definition and Theorem 4.25.

$$\hat{\pi}(X \vee Y) = \frac{1}{\lambda^{n-1} \det \hat{\pi}} \cdot \hat{\pi}(X) \wedge \hat{\pi}(Y), \tag{1.16}$$

$$\hat{\rho}(X \vee Y) = \mu \cdot \hat{\rho}(X) \wedge \hat{\rho}(Y), \tag{1.17}$$

$$\hat{\rho}(X \wedge Y) = \frac{1}{\mu^{n-1} \det \hat{\rho}} \cdot \hat{\rho}(X) \vee \hat{\rho}(Y). \tag{1.18}$$

The preservation of the two exterior products leads to the conclusion, that projective transformations always preserve incidence, which is a basic fact in projective geometry.

Section 4 closes by looking at the group properties of the algebra homomorphisms.

Projective geometry $\mathcal{P}_n$ is described in terms of projective algebra Λ_n in Sections 5 to 7. Section 5 provides a system of axioms for projective geometry $\mathcal{P}_n$ in terms of projective algebra Λ_n as well as fundamental concepts like the principle of duality, primitive geometric forms, general position of basic elements, cross ratio of four different basic elements and several examples.

A system of axioms is much more than the definition of an $(n-1)$-dimensional projective space $\mathbb{F}P^{(n-1)}$ with respect to the field $\mathbb{F}$. A projective space $\mathbb{F}P^{(n-1)}$ comprises all the 1-dimensional subspaces of the n-dimensional vector space $\mathbb{F}^n$ in order to represent the basic geometric elements of the former. E. g., the 1-dimensional subspaces of the vector space $\mathbb{F}^3$ may represent all the points of the projective planar field $\mathbb{F}P^2$ or all the lines of the projective planar field $\mathbb{F}P^2$ or all the lines of the projective centric bundle $\mathbb{F}P^2$ or all the planes of the projective centric bundle $\mathbb{F}P^2$ and the 1-dimensional subspaces of the vector space $\mathbb{F}^4$ may represent all the points of projective space $\mathbb{F}P^3$ or all the planes of projective space $\mathbb{F}P^3$ and the 1-dimensional subspaces of the vector space $\mathbb{F}^6$ may represent all the linear complexes including the lines of projective space $\mathbb{F}P^5$.

The system of axioms given in Definition 5.3 has three parts, in the first one of which the $n+1$ different types of basic geometric elements are introduced. The $n+1$ different types of basic elements correspond to the $n+1$ different vector subspaces Λ_n^k of projective algebra Λ_n,

$$\begin{aligned} \Lambda_n &= \Lambda_n^{0+} \oplus \Lambda_n^{1+} \oplus \cdots \oplus \Lambda_n^{(n-1)+} \oplus \Lambda_n^{n+} \\ &= \Lambda_n^{n-} \oplus \Lambda_n^{(n-1)-} \oplus \cdots \oplus \Lambda_n^{1-} \oplus \Lambda_n^{0-}, \end{aligned} \tag{1.19}$$

with

$$\Lambda_n^{k+} = \Lambda_n^{(n-k)-}, \qquad \dim \Lambda_n^{k+} = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda_n^{(n-k)-}. \tag{1.20}$$

Each of the $n+1$ different $\binom{n}{k}$-dimensional vector subspaces Λ_n^k represents an $[\binom{n}{k} - 1]$-dimensional projective space.

As an example let us take projective geometry in space $\mathcal{P}_4$ with $n = 4$. The five different basic elements in this case are: the space of planes as such belonging to the projective space $\Lambda_4^{0+} = \Lambda_4^{4-}$, the points belonging to the projective space $\Lambda_4^{1+} = \Lambda_4^{3-}$, the linear complexes (including the lines) belonging to the projective space $\Lambda_4^{2+} = \Lambda_4^{2-}$, the planes belonging to the projective space $\Lambda_4^{3+} = \Lambda_4^{1-}$ and the space of points as such belonging to the projective space $\Lambda_4^{4+} = \Lambda_4^{0-}$.

Part two of the system of axioms declares the incidence relations between generic geometric elements and part three the geometric operations of connection and intersection. Thus, the system of axioms is a set of rules, which completely defines projective geometry $\mathcal{P}_n$ in any dimension n. It represents analytically into every detail the wealth of synthetic projective geometry as it has been described, e. g., in the books by LOUIS LOCHER-ERNST. [Loc80b, Loc80a, LE70]

Let us emphasise the difference of how usually *join* and *meet* are understood and how in terms of projective algebra the operations of connection and intersection are defined. Usually, *join* and *meet* are defined — also in the context of projective geometry — in terms of order theory as *supremum* (least upper bound) and *infimum* (greatest lower bound) respectively. It leads in several cases to the same result as in the approach here with projective algebra, but not in general. This is also why we are not using the terms 'join' and 'meet' but instead 'operation of connection' and 'operation of intersection' or just 'connect' and 'intersect' respectively. In the planar field $\mathcal{P}_3$ e. g., join and meet as well as the operation of connection and the operation of intersection are the same respectively. Especially two different lines (including the parallel ones) intersect in one and exactly one point. If we go to spacial projective geometry $\mathcal{P}_4$, the meet of two different lines is void, in case the two lines are skew, or, else, it delivers the meeting point. In the approach with projective algebra two skew lines intersect in a non-zero 4-vector in the minus approach or, else, the operation of intersection between two different, but non-skew lines results in the zero-vector. Nevertheless, in the approach with projective algebra, there exists one and exactly one point in projective space, which is incident with the two different, but non-skew lines. It is the same point, which the operation of meet would deliver as meeting point.

So, what is the advantage of defining the operations of connection and intersection in terms of projective algebra, as it is done in Definition 5.3 of 2^n-dimensional projective geometry $\mathcal{P}_n$? The properties of the major and minor exterior products $\wedge$ and $\vee$ are clearly given by Definition 4.2 of the 2^n-dimensional unity free exterior algebra Λ_n. And the geometric operations of connection and intersection follow in terms of projective algebra by part three of Definition 5.3. Let us illustrate the advantage of this approach with projective algebra with spacial projective geometry $\mathcal{P}_4$: The operations of connection and intersection then are not only valid for points and planes, but also for lines and linear complexes. They make sense for all basic elements of spacial projective geometry $\mathcal{P}_4$.

At the very latest, when we proceed to the incidence relations of part two in the Definition 5.3 of spacial projective geometry $\mathcal{P}_4$, this advantage becomes obvious. The incidence for two generic geometric elements is defined in equation (5.3) in terms of the two vanishing exterior products. This definition integrates the incidence relations for points and planes — which is equivalent to the order theoretical definition of join and meet — as well as the incidence relations for lines[6] and the incidence relations for linear complexes.[7] One and the same definition of what incidence is — compare equation (5.3) — holds for all involved basic elements.

Projective geometry is a non-metric geometry. Nevertheless, there are non-metric quantities, which are preserved under the operations of connection and intersection or under projective transformations. A well known quantity of this kind is the cross ratio of four different basic elements. The cross ratio is introduced in the non-metric context of projective algebra Λ_n in Definition 5.15. Several examples conclude Section 5.

In Section 6 we introduce projective transformations on the basis of algebra isomorphisms, show that the projectivities among the latter form a group with respect to concatenation and proof the fundamental theorem of projective geometry. Section 6 concludes by applying the concept of projective transformations in the planar field $\mathcal{P}_3$ in four different examples.

In Section 7 we review and provide the concepts of polarities, quadratic and bilinear forms, quadrics, and orthonormal bases in the context of projective geometry $\mathcal{P}_n$. In addition, we introduce, what a pair of naturally associated — non-degenerate and degenerate — polarities is. We define harmonic orthonormal systems of bases and list the different types of quadrics in $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$.

The article concludes in Section 8 with the transition from projective algebra Λ_n to Clifford double algebra Γ_n and to the metric Cayley-Klein geometries. Definition 8.5 introduces the major Clifford product (no sign) and the minor Clifford product ($*$), Theorem 8.7 provides the multiplication tables for these Clifford products in any dimension n and the two inner products are given in terms of the respective Clifford products in Definition 8.15. Theorem 8.20 records the result: Graßmann algebras are most degenerate Clifford algebras and the unital property is the last remnant of the Clifford product. For all non-zero grades the Clifford product and the respective exterior product are the same.

In Definition 8.22 we describe the metric version of space and counterspace. And in the last subsection of this article we compare the euclidean PGA's by Charles Gunn and by Leo Dorst and Steven De Keninck with the euclidean Clifford double algebra $\Gamma_{1,0,3;3,0,1}$. The later closes a gap between euclidean geometry and its representation by the euclidean PGA's. With the second Clifford product also the point-wise signature $(1,0,3)$ is available directly. See Subsection 8.5 on metric geometries in general and euclidean and dual euclidean geometry in particular.

Euclidean geometry $\Gamma_{1,0,3;3,0,1}$ and dual euclidean geometry $\Gamma_{3,0,1;1,0,3}$ are taken to illustrate the concept of metric space and of metric counterspace.

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[6] See Chapter 7 in [Sto95, p. 106].

[7] Compare the Definition 3.20 in [Sto99, p. 106].

Acknowledgments

The ideas, concepts and results presented in this article were developed in a longer course of time. It was important to speak about the evolving ideas in various contexts, such as within the work and meetings of the

- Mathematics Colloquium at the Section for Mathematics and Astronomy of the Goetheanum in Dornach, Switzerland.

The latter has been running three times a year for over 20 years and the meetings took place in Dornach (Switzerland) and in Driebergen-Zeist (The Netherlands). Among others, there were contributions by Gerhard Kowol [Kow09], Nick Thomas [Tho08], Hanns-Jörg Stoss [Sto95, Sto99, Sto09], Lou de Boer [dB09, dB23, dB24], Charles Gunn [Gun11a, Gun11b, Gun17], Thomas Neukirchner [ACE$^+$25], Nadine Braxmeier-Even, Russell Arnold [ACE$^+$25], Immo Diener [Die17, Die21], Morten Eide. [ACE$^+$25, Eid96] And we studied books by Peter Gschwind [Gsc91b, Gsc91a, Gsc00], by Renatus Ziegler [Zie85, Zie12], by George Adams [Ada65, Ada81, Ada96] and by Henry Frederick Baker [Bak15].

Other occasions to speak about the evolving ideas were the following Clifford Algebra Conferences:

- 8th International Conference on Clifford Algebras and their Applications in Mathematical Physics (ICCA8) from May 26-30, 2008 in Campinas, Sao Paulo, Brazil;
- 10th International Conference on Clifford Algebras and their Applications in Mathematical Physics (ICCA10) from August 4–9, 2014 at the University of Tartu, Estonia;
- Alterman Conference on Geometric Algebra and Summer School on Kähler Calculus from August 1-9, 2016 at the at the Faculty of Mathematics and Informatics of the Transilvania University of Brasov, Romania; and
- 3rd International Conference on Mathematical Methods in Physics from April 27-30, 2026, Marrakech, Morocco (ICMMP3).

Many thanks go to all the participants and speakers of these meetings. Without their inputs and feedbacks, this work would not have been possible.

The final reason for this article is the ongoing research project ‘Efficasy of the Smallest Particles and Substance Forces in Space and Counterspace’. This project is being carried out in cooperation with the experimental research group of Prof. Dr. Stephan Baumgartner from the Society for Cancer Research in Arlesheim, Switzerland. We would like to sincerely thank the Software AG Stiftung in Darmstadt, Germany, which has provided the generous funding for this project to the Section for Mathematics and Astronomy at the Goetheanum for the last three years.

2. Complementary Graded Vector Spaces and Algebras

Definition 2.1 (Multi Vectors, Homogeneous Multi Vectors, k-Vectors, Vectors of grade k). Let $\mathcal{Z}_{n+1}$ denote the additive group of the factor ring $\mathbb{Z}/(n+1)\mathbb{Z}$ with $n \in \mathbb{N}$. And let V be a $\mathcal{Z}_{n+1}$-graded vector space (cf. [Gre81, pp. 167]) with the direct decomposition

$$V = \bigoplus_{k=0}^{n} V_k. \tag{2.1}$$

The elements $M \in V$ are called *(generic) multi vectors*, the elements $X \in V_k$ *homogeneous multi vectors* or *k-vectors* or *vectors of grade k*. In order to indicate the grade, a k-vector X may be endowed with an overlined subscript k,

$$X = X_{\bar{k}}. \tag{2.2}$$

Definition 2.2 (k-Projection). Let $V = \bigoplus_{k=0}^{n} V_k$ be a $\mathcal{Z}_{n+1}$-graded vector space and $k \in \mathbb{N}$ with $0 \leq k \leq n$. The mapping

$$\begin{array}{rccc} \langle\ \ \rangle_k : & V & \longrightarrow & V_k \\ & X = \displaystyle\sum_{l=0}^{n} X_{\bar{l}} & \longmapsto & \langle X \rangle_k := X_{\bar{k}} \end{array} \tag{2.3}$$

is called a *k-projection* and denoted by angle brackets with subscript k.

Lemma 2.3. *Let $V = \bigoplus_{k=0}^{n} V_k^+$ be a $\mathcal{Z}_{n+1}$-graded vector space. Then there is one and only one second $\mathcal{Z}_{n+1}$-grading $(V_k^-)_{k \in \mathcal{Z}_{n+1}}$ for the same vector space V with the condition*

$$V_k^- = V_{n-k}^+ \qquad \forall\ k \in \mathcal{Z}_{n+1}. \tag{2.4}$$

In the limiting case $n = 0$ both $\mathcal{Z}_{n+1}$-gradings $(V_k^+)_{k \in \mathcal{Z}_{n+1}}$ and $(V_k^-)_{k \in \mathcal{Z}_{n+1}}$ coincide, otherwise they are different.

Proof. For $n \neq 0$ we always have $V_0^- = V_n^+ \neq V_0^+$. Thus, the two $\mathcal{Z}_{n+1}$-gradings $(V_k^+)_{k \in \mathcal{Z}_{n+1}}$ and $(V_k^-)_{k \in \mathcal{Z}_{n+1}}$ are for $n \neq 0$ different. □

Definition 2.4 (Complementary Graded Vector Space, Plus Approach, Minus Approach). A vector space V is called *complementary graded*, if and only if it is endowed with the two $\mathcal{Z}_{n+1}$-gradings $(V_k^+)_{k \in \mathcal{Z}_{n+1}}$ and $(V_k^-)_{k \in \mathcal{Z}_{n+1}}$ according to Lemma 2.3. The decomposition

$$V = \bigoplus_{k=0}^{n} V_k^+ \tag{2.5}$$

is called *plus approach of V* and the decomposition

$$V = \bigoplus_{k=0}^{n} V_k^- \tag{2.6}$$

is called *minus approach of V*.

Example. In the case $n = 4$ we have

$$\begin{aligned} V &= V_0^+ \oplus V_1^+ \oplus V_2^+ \oplus V_3^+ \oplus V_4^+ \\ &= V_4^- \oplus V_3^- \oplus V_2^- \oplus V_1^- \oplus V_0^- \end{aligned}$$

with $V_0^+ = V_4^-$, $V_1^+ = V_3^-$, $V_2^+ = V_2^-$, $V_3^+ = V_1^-$ and $V_4^+ = V_0^-$.

Notation 2.5 (Plus Minus Notation). The vectors and subsets of a complementary graded vector space V may be supplied with a superscript plus sign in order to indicate that they belong to the plus approach (*plus notation*) or with a superscript minus sign in order to indicate that they belong to the minus approach (*minus notation*).

The plus minus notation is *optional.* Out of the context it is often already clear in which approach one is working. We do not need to supply the sets and multi vectors in this case with an extra superscript plus or minus sign. The plus minus notation can also be left out in the case a certain expression is valid in both approaches.

Example. Let V be a complementary graded vector space. Then for a homogeneous multi vector $X \in V_k^+ = V_{n-k}^-$ we may write

$$X^+_{\overline{k}} = X = X^-_{\overline{n-k}}. \tag{2.7}$$

Example. Let $P_i = \langle P_i \rangle_1^+$ denote n 1-vectors in the plus approach and $E_i = \langle E_i \rangle_1^-$ n 1-vectors in the minus approach. The sets of these homogeneous multi vectors,

$$\mathcal{P}^+ = \{P_i^+ \mid i \in \mathbb{N},\ 1 \le i \le n\}, \tag{2.8}$$

$$\mathcal{E}^- = \{E_i^- \mid i \in \mathbb{N},\ 1 \le i \le n\}, \tag{2.9}$$

may then be supplied by a superscript plus or minus sign respectively.

Let $\mathcal{A}(+,\cdot,\ )$ denote an algebra. According to [Gre81, pp. 174] $\mathcal{A}$ is called a $\mathcal{Z}_{n+1}$-graded algebra if and only if

(a) the vector space $\mathcal{A}(+,\cdot)$ is $\mathcal{Z}_{n+1}$-graded and
(b) for any two homogeneous multi vectors $X_{\overline{r}}$ and $Y_{\overline{s}}$ the product

$$X_{\overline{r}} Y_{\overline{s}} = \langle X_{\overline{r}} Y_{\overline{s}} \rangle_{r+s} \tag{2.10}$$

is again homogeneous of grade $r + s$.

With respect to the $\mathcal{Z}_{n+1}$-grading of the vector space $\mathcal{A}(+,\cdot)$ in the minus approach — cf. Definition 2.4 and Lemma 2.3 — equation (2.10) reads

$$X^-_{\overline{n-r}} Y^-_{\overline{n-s}} = \langle X^-_{\overline{n-r}} Y^-_{\overline{n-s}} \rangle^-_{n-(r+s)}, \tag{2.11}$$

i. e. the grades do not add in the minus approach (for $n \neq 0$).

In order to get a *complementary* graded algebra, the algebra $\mathcal{A}$ should be $\mathcal{Z}_{n+1}$-graded with respect to the plus as well as to minus approach of the vector space $\mathcal{A}(+,\cdot)$. In general, this is not possible with only one algebra multiplication. We therefore have

Definition 2.6 (Complementary Graded Algebra). Let the vector space $\mathcal{A}(+,\cdot)$ form an algebra with respect to the product noted by juxtaposition and a second algebra with respect to the product noted by the symbol $*$. The double algebra $\mathcal{A}(+,\cdot,\ ,*)$ is called a *complementary graded algebra* if and only if

(a) $\mathcal{A}(+,\cdot)$ is a complementary graded vector space,
(b) $\mathcal{A}(+,\cdot,\ )$ is a $\mathcal{Z}_{n+1}$-graded algebra with respect to the plus approach and
(c) $\mathcal{A}(+,\cdot,*)$ is a $\mathcal{Z}_{n+1}$-graded algebra with respect to the minus approach.

3. Binary Indices

Other authors have already used binary numbers to label elements of Clifford algebras in various ways. See [Mar04] or [Lou01, pp. 279-287]. We adopted the main idea and shortly review binary numbers and several binary operators in this section.

Notation 3.1 (m-Digit Binary Numbers). In order to label basis elements and other homogeneous multi vectors we introduce binary numbers

$$\mathbf{b} \equiv b_{m-1}\dots b_1 b_0 = \left[\sum_{k=0}^{m-1} b_k 2^k\right]_{10}, \quad m \in \mathbb{N}, \quad b_k \in \{0,1\}, \tag{3.1}$$

with m digits $b_{m-1}, \dots, b_0$. The binary numbers will be noted by small latin letters in bold face, its m digits with the same letter but normal face and the digits will be numbered from 0 to $m-1$.

A real m-digit binary number $\mathbf{b}$ translated into a real decimal number lies between $[0]_{10}$ and $[2^m-1]_{10}$.

Definition 3.2 (Binary Check Sum). The sum $S(\mathbf{b})$ of the digits of the binary number (3.1) is

$$S(\mathbf{b}) := \left[\sum_{k=0}^{n-1} b_k\right]_{10}. \tag{3.2}$$

Remark. The m-digit binary number $\mathbf{b}$

(a) is zero, if and only if $S(\mathbf{b}) = 0$;
(b) translates into a decimal power of two, if and only if $S(\mathbf{b}) = 1$;
(c) translates into the decimal number $[2^m-1]_{10}$, if and only if $S(\mathbf{b}) = m$.

Proof. (a) $S(\mathbf{b}) = 0 \Leftrightarrow b_i = 0 \;\forall\; 0 \le i \le m-1 \Leftrightarrow \mathbf{b} = [0]_2 = [0]_{10}$.

(b) We have $S(\mathbf{b}) = 1$ if and only if there is one index $l \in \mathbb{N}$, such that the binary number

$$\begin{aligned} \mathbf{b} &= b_{m-1}\dots b_1 b_0 \qquad & b_i &= \begin{cases} 1, & \text{if } i = l, \\ 0, & \text{if } i \neq l, \end{cases} \\ &= [2^l]_{10}, & & 0 \le l \le m-1, \end{aligned} \tag{3.3}$$

translates into a decimal power of two.

(c) $S(\mathbf{b}) = m \Leftrightarrow b_i = 1 \ \forall \ 0 \leq i \leq m-1 \Leftrightarrow \mathbf{b} = [2^m - 1]_{10}$.

$\square$

Definition 3.3 (Binary Complement). The binary complement of a m-digit binary number $\mathbf{b}$ is

$$\overline{\mathbf{b}} = \overline{b_{m-1} \ldots b_1 b_0} := \left[\sum_{k=0}^{m-1} \overline{b}_k 2^k \right]_{10} \qquad \text{with} \qquad \overline{0} := 1, \qquad \overline{1} := 0. \tag{3.4}$$

The binary complement translated into a decimal number lies again between $[0]_{10}$ and $[2^n - 1]_{10}$. In addition we always have

$$\mathbf{b} + \overline{\mathbf{b}} = \mathbf{u} = [2^m - 1]_{10}, \tag{3.5}$$

$$\overline{\overline{\mathbf{b}}} = \mathbf{b}, \tag{3.6}$$

where $\mathbf{u}$ is the m-digit binary number with $S(\mathbf{u}) = m$.

Definition 3.4 (Indices on the Left Side). Let $\mathbf{b}$ be a m-digit binary number and $\mathcal{I} := \{i_1, \ldots, i_k\}$ an ordered index set of length k with

$$0 \leq i_1 < i_2 < \ldots < i_k \leq m-1, \qquad k, j, i_j \in \mathbb{N}, \qquad 1 \leq j \leq k, \tag{3.7}$$

such that

$$\mathbf{b} = b_{m-1} \ldots b_1 b_0, \qquad\qquad b_i = \begin{cases} 1, & \text{if } i \in \mathcal{I}, \\ 0, & \text{if } i \notin \mathcal{I}. \end{cases} \tag{3.8}$$

In addition, let l_1, l_2, …, l_p be p natural numbers with

$$1 \leq l_1 < l_2 \ldots < l_p \leq k, \qquad l_j \in \mathbb{N}, \qquad 1 \leq j \leq p. \tag{3.9}$$

These numbers determine a subset of $\mathcal{I}$,

$$\mathcal{L} := \{i_{l_1}, \ldots, i_{l_p}\} \subset \mathcal{I}, \tag{3.10}$$

and thus establish an ordered index subset $\mathcal{L}$ of length p.

The natural numbers l_1, l_2, …, l_p represent as *lower left indices in brackets* the binary number

$$\begin{aligned} {}_{(l_1 l_2 \ldots l_p)}\mathbf{b} &:= \mathbf{c} & (3.11) \\ &= c_{m-1} \ldots c_1 c_0, \qquad c_i = \begin{cases} 1, & \text{if } i \in \mathcal{L}, \\ 0, & \text{if } i \notin \mathcal{L}, \end{cases} \end{aligned}$$

and as *upper left indices in brackets* the binary number

$$ {}^{(l_1 l_2 \ldots l_p)}\mathbf{b} := \overline{{}_{(l_1 l_2 \ldots l_p)}\mathbf{b}}. \tag{3.12}$$

The brackets may be omitted, if there is only one lower left or upper left index.

Example. As an example for the left indices let us take the number $\mathbf{b} = [01011]_2$. Then the index set $\mathcal{I}$ is $\{0, 1, 3\}$, i.e. $i_1 = 0$, $i_2 = 1$ und $i_3 = 3$ and we have:

$$S(\mathbf{b}) = 3 \qquad\qquad S\big(\overline{\mathbf{b}}\big) = 2$$

$$\mathbf{b} = 01011 \qquad\qquad \overline{\mathbf{b}} = 10100$$

$$\begin{array}{rlrlrl}
{}_1\mathbf{b} &= 00001 & {}^1\mathbf{b} &= 11110 & {}_1\overline{\mathbf{b}} &= 00100 \\
{}_2\mathbf{b} &= 00010 & {}^2\mathbf{b} &= 11101 & {}_2\overline{\mathbf{b}} &= 10000 \\
{}_3\mathbf{b} &= 01000 & {}^3\mathbf{b} &= 10111 & {}_{(12)}\overline{\mathbf{b}} &= 10100 = \overline{\mathbf{b}} \\
{}_{(12)}\mathbf{b} &= 00011 & {}^{(12)}\mathbf{b} &= 11100 & & \\
{}_{(13)}\mathbf{b} &= 01001 & {}^{(13)}\mathbf{b} &= 10110 & {}^1\overline{\mathbf{b}} &= 11011 \\
{}_{(23)}\mathbf{b} &= 01010 & {}^{(23)}\mathbf{b} &= 10101 & {}^2\overline{\mathbf{b}} &= 01111 \\
{}_{(123)}\mathbf{b} &= 01011 = \mathbf{b} & {}^{(123)}\mathbf{b} &= 10100 = \overline{\mathbf{b}} & {}^{(12)}\overline{\mathbf{b}} &= 01011 = \mathbf{b}
\end{array}$$

Remark. For any m-digit binary number $\mathbf{b} = b_{m-1} \dots b_1 b_0$ with check sum $S(\mathbf{b}) = k$, $0 < k \leq m$, and

$$\mathbf{b} = \sum_{l=1}^{k} {}_l\mathbf{b}, \tag{3.13}$$

we always have

$${}_l\mathbf{b} < {}_{l+1}\mathbf{b}, \tag{3.14}$$

$$S({}_l\mathbf{b}) = 1. \tag{3.15}$$

Proof. By precondition there are exactly k indices

$$0 \leq i_1 < i_2 < \dots < i_k \leq m-1, \tag{3.16}$$

the binary digits of which do not vanish

$$b_i = \begin{cases} 1, & i \in \{i_1, \dots, i_k\}, \\ 0, & i \notin \{i_1, \dots, i_k\}. \end{cases} \tag{3.17}$$

Thus we have

$$\mathbf{b} = b_{m-1} \dots b_1 b_0 = \left[\sum_{l=1}^{k} 2^{i_l} \right]_{10} = \sum_{l=1}^{k} {}_l\mathbf{b} \tag{3.18}$$

with ${}_l\mathbf{b} = [2^{i_l}]_{10}$ and ${}_{l+1}\mathbf{b} = [2^{i_{l+1}}]_{10}$., i. e. ${}_l\mathbf{b} < {}_{l+1}\mathbf{b}$. The equation $S({}_l\mathbf{b}) = 1$ is a consequence of Remark 3. □

Notation 3.5 (Nested Sequence of Left Indices). For a binary number with several left indices, the nested sequence of left indices has to be read from inside outwards as in the following example,

$${}^{\;l}_{4\,(23)}\mathbf{b} = {}_4\left[{}^l \left[{}_{(23)}\mathbf{b} \right] \right]. \tag{3.19}$$

Remark. Let V be a vector space, $A_i \in V$, $i \in \{1, \dots, m\} \subset \mathbb{N}$ and $\mathbf{b}$ a binary variable with m digits. We may then label the m elements A_i with

$$V \ni A_{\mathbf{b}}, \qquad S(\mathbf{b}) = 1, \tag{3.20}$$

i. e. we use all binary numbers with $S(\mathbf{b}) = 1$ to label the vectors $A_{\mathbf{b}}$.

Proof. With $\mathbf{b}_i = \left[2^{i-1}\right]_{10}$ we get $S(\mathbf{b}_i) = 1$ and $A_i = A_{\mathbf{b}_i}$, $1 \leq i \leq m$. □

In this article we will also use the bitwise binary operators AND and XOR. Example: 1001 AND 0101 = 0001 and 1001 XOR 0101 = 1100.

4. Unity Free Exterior Double Algebra Λ_n

Notation 4.1 (Multiple Exterior Product Signs). The two exterior products of the unity free exterior double algebra Λ_n are denoted by $\wedge$ and $\vee$. For multiple exterior products we use with $l, m \in \mathbb{N} \setminus \{0\}$

$$\bigwedge_{l=1}^{m} X_l := X_1 \wedge X_2 \wedge \cdots \wedge X_m, \qquad \bigvee_{l=1}^{m} X_l := X_1 \vee X_2 \vee \cdots \vee X_m. \tag{4.1}$$

Definition 4.2 (Unity Free Exterior Double Algebra Λ_n). A *unity free exterior double* $\mathbb{F}$*-algebra* $\Lambda_n(+,\cdot,\wedge,\vee)$, or short *exterior double algebra*, is a set Λ_n with four operations:

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{+} & \Lambda_n \\ (A,B) & \longmapsto & A+B \end{array} \qquad \begin{array}{ccc} \mathbb{F} \times \Lambda_n & \xrightarrow{\cdot} & \Lambda_n \\ (\alpha,A) & \longmapsto & \alpha \cdot A \end{array} \tag{4.2}$$

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{\wedge} & \Lambda_n \\ (A,B) & \longmapsto & A \wedge B \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{\vee} & \Lambda_n \\ (A,B) & \longmapsto & A \vee B \end{array} \tag{4.3}$$

The operations are called addition ($+$), scalar multiplication (no sign or $\cdot$), major exterior product ($\wedge$) and minor exterior product ($\vee$). The four operations obey the following conditions:

(P1) $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$.

(P2) $\Lambda_n(+,\cdot)$ is a complementary graded $\mathbb{F}$-vector space of dimension 2^n

$$\Lambda_n(+,\cdot) = \bigoplus_{k=0}^{n} \Lambda_n^{k+}(+,\cdot) = \bigoplus_{k=0}^{n} \Lambda_n^{k-}(+,\cdot), \qquad k, n \in \mathbb{N}, \tag{4.4}$$

with the dimensions

$$\dim\left(\Lambda_n^k(+,\cdot)\right) = \binom{n}{k}, \qquad 0 \le k \le n, \tag{4.5}$$

for the subspaces.

(P3) $\Lambda_n(+,\cdot,\wedge)$ and $\Lambda_n(+,\cdot,\vee)$ are two associative $\mathbb{F}$-algebras without identity element. In addition, both exterior products live up to the requirements:

- All scalars $X_{\overline{0}} \in \Lambda_n^0(+,\cdot)$ are left and right zero divisors,

$$X_{\overline{0}}^{+} \wedge M = M \wedge X_{\overline{0}}^{+} = \mathbf{0}, \qquad \forall\, M \in \Lambda_n(+,\cdot), \tag{4.6}$$

$$X_{\overline{0}}^{-} \vee M = M \vee X_{\overline{0}}^{-} = \mathbf{0}, \qquad \forall\, M \in \Lambda_n(+,\cdot). \tag{4.7}$$

- Exterior products between homogeneous multi vectors add the grades,

$$A_{\overline{r}}^{+} \wedge B_{\overline{s}}^{+} = \langle A_{\overline{r}}^{+} \wedge B_{\overline{s}}^{+} \rangle_{r+s}^{+}, \qquad r+s \le n, \tag{4.8}$$

$$A_{\overline{r}}^{-} \vee B_{\overline{s}}^{-} = \langle A_{\overline{r}}^{-} \vee B_{\overline{s}}^{-} \rangle_{r+s}^{-}, \qquad r+s \le n. \tag{4.9}$$

- For 1-vectors $A_i \in \Lambda_n^{1+}$ or $B_i \in \Lambda_n^{1-}$ we have with $l > 1$

$$\bigwedge_{i=1}^{l} A_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} A_1,\ A_2,\ \ldots,\ A_l \text{ are} \\ \text{linearly dependent.} \end{array} \right. \tag{4.10}$$

$$\bigvee_{i=1}^{l} B_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} B_1,\ B_2,\ \ldots,\ B_l \text{ are} \\ \text{linearly dependent.} \end{array} \right. \tag{4.11}$$

Remark. The equations (4.8) and (4.9) include the scalars too ($r = 0$ or $s = 0$), since $\mathbf{0}$ belongs to any direct subspace Λ_n^{r+s} of the exterior double algebra Λ_n.

Theorem 4.3. *The unity free exterior double algebra is a complementary graded algebra in the sense of Definition 2.6.*

Proof. By Axiom (P2) the vector space $\Lambda_n(+,\cdot)$ is complementary graded. By equations (4.8) and (4.9) of Axiom (P3) the algebras $\Lambda_n(+,\cdot,\wedge)$ and $\Lambda_n(+,\cdot,\vee)$ respectively are $\mathscr{Z}_{n+1}$-graded. □

Notation 4.4 (Combined Exterior Product). Any mathematical term which contains the combined exterior product $\Diamond$ can be read twice: Firstly with respect to the plus approach as the major exterior product $\wedge$ and secondly with respect to the minus approach as the minor exterior product $\vee$.

Example. The expression

$$X_{\bar{1}} \Diamond Y_{\bar{1}} = -Y_{\bar{1}} \Diamond X_{\bar{1}} \qquad \forall\ X_{\bar{1}}, Y_{\bar{1}} \in \Lambda_n^1 \tag{4.12}$$

means

$$X_{\bar{1}}^{+} \wedge Y_{\bar{1}}^{+} = -Y_{\bar{1}}^{+} \wedge X_{\bar{1}}^{+} \qquad \forall\ X_{\bar{1}}^{+}, Y_{\bar{1}}^{+} \in \Lambda_n^{1+} \tag{4.13}$$

$$X_{\bar{1}}^{-} \vee Y_{\bar{1}}^{-} = -Y_{\bar{1}}^{-} \vee X_{\bar{1}}^{-} \qquad \forall\ X_{\bar{1}}^{-}, Y_{\bar{1}}^{-} \in \Lambda_n^{1-} \tag{4.14}$$

and is a consequence of equations (4.10) and (4.11). Since $X_{\bar{1}} + Y_{\bar{1}}$ is linearly dependent to itself, we have $(X_{\bar{1}} + Y_{\bar{1}}) \Diamond (X_{\bar{1}} + Y_{\bar{1}}) = X_{\bar{1}} \Diamond Y_{\bar{1}} + Y_{\bar{1}} \Diamond X_{\bar{1}} = 0$, i. e. the exterior products are antisymmetric both.

Notation 4.5 (Multiple Combined Exterior Product Sign). Analogous to the notation for multiple exterior products we use for multiple combined exterior products the sign

$$\mathop{\Diamond}_{l=1}^{m} X_l := X_1 \Diamond X_2 \Diamond \cdots \Diamond X_m. \tag{4.15}$$

The unity free exterior double algebra and Graßmann algebra look very similar. Let us shortly compare the two structures. A Graßmann algebra $\bigwedge V$ of a vector space V with $\dim V = n$ is an associative, unital, graded and antisymmetric algebra of dimension 2^n. Unity free exterior double algebra Λ_n is a complementary graded algebra, this is why we can only check whether the plus approach $\Lambda_n(+,\cdot,\wedge)$ or the minus approach $\Lambda_n(+,\cdot,\vee)$ shows the structure of a Graßmann algebra.

There is no difference between the algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ and $\bigwedge V$ inasmuch as they are all associative, graded, antisymmetric and inasmuch as they have the same dimensions on the level of the whole algebra as well as on the level of their direct subspaces. The difference between the algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ on one side and $\bigwedge V$ on the other side is that there is no identity element present in the unity free exterior algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ — all scalars are zero divisors — and the Graßmann algebra $\bigwedge V$ is unital.

Remark. Why don't we introduce an own notation for the exterior products in the unity free double algebras Λ_n? The exterior products in the unity free double algebras Λ_n are different from the product of the usual Graßmann algebras $\bigwedge V$ (as well as from $\bigwedge V^*$) with respect to the scalars, and the same with respect to all other homogeneous k-vectors. Here are the reasons for keeping the notation of the Graßmann algebras also in the context of the unity free exterior double algebras:

- In most applications of the exterior product from Graßmann algebra the property
  $$\mathbf{1}\wedge A = A\wedge\mathbf{1} = A \qquad \forall\, A\in\bigwedge V$$
  is not used. This is why we can exchange this rule as we did in equations (4.6) and (4.7) of Definition 4.2 without effecting most of the applications.
- In projective geometry, incidence is defined in terms of the two exterior products. Compare equation (5.3) of Definition 5.3. In most applications scalars are not involved when we look at the incidence of two geometric elements. But if they are involved, out of geometric reasons, the scalars are zero divisors with respect to both exterior products.
- In Section 8 we will do the transition from the unity free exterior double algebras Λ_n to Clifford double algebras Γ_n, i. e., we introduce in the context of the unity free exterior double algebras $\Lambda_n(+,\cdot,\wedge,\vee)$ two different Clifford products denoted by juxtaposition and by the symbol $*$. The double algebras $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ continue to have no identities with respect to the exterior products $\wedge$ and $\vee$, while the two Clifford products are both unital (with respect to the identities $\mathbf{1}^+ = \mathbf{Z}^+$ and $\mathbf{1}^- = \mathbf{Z}^-/\lambda$). If now both Clifford algebras of the double algebra Γ_n degenerate as much as possible, i. e. the algebras $\Gamma_n^+(+,\cdot,\ )$ and $\Gamma_n^-(+,\cdot,*)$ both carry signature $(0,0,n)$ and the parameters λ, μ satisfy $\lambda=\mu=1$, then these algebras represent both ordinary Graßmann algebras $\bigwedge V$. In both algebras $\Gamma^+_{(0,0,n)}$ and $\Gamma^-_{(0,0,n)}$ from the respective Clifford product only the identity element remains,
  $$\mathbf{1}^+A = A\mathbf{1}^+ = A \qquad \forall\, A\in\Gamma^+_{(0,0,n)},$$

$$\mathbf{1}^- * A = A * \mathbf{1}^- = A \qquad \forall\, A \in \Gamma^-_{(0,0,n)}.$$

Compare Theorem 8.20.

- Our treatment shows:
  (a) It is natural to see the scalars with respect to an exterior product as zero divisors.
  (b) Graßmann algebras are most degenerate Clifford algebras and the unital property is the last remnant of the Clifford product and not a property of the exterior product.

Definition and Theorem 4.6 (Basis in the Plus Approach). Let $\mathbf{b}$ be a binary variable with n digits, $\{P_{\mathbf{b}}\}$ with $S(\mathbf{b}) = 1$ a set of n basis 1-vectors from Λ_n^{1+} and $\mathbf{Z}^+ \in \Lambda_n^{0+} \setminus \{\mathbf{0}\}$ a vector of grade 0. Then the homogeneous multi vectors

$$P_{\mathbf{b}} := \begin{cases} \mathbf{Z}^+, & S(\mathbf{b}) = 0, \\ \bigwedge_{l=1}^{S(\mathbf{b})} P_{{}_l\mathbf{b}}, & 0 < S(\mathbf{b}) \leq n, \end{cases} \tag{4.16}$$

form a basis for the 2^n-dimensional vector space of exterior double algebra Λ_n.

Definition and Theorem 4.7 (Basis in the Minus Approach). Let $\mathbf{b}$ be a binary variable with n digits, $\{E_{\mathbf{b}}\}$ with $S(\mathbf{b}) = 1$ a set of n basis 1-vectors from Λ_n^{1-} and $\mathbf{Z}^- \in \Lambda_n^{0-} \setminus \{\mathbf{0}\}$ a vector of grade 0. Then the homogeneous multi vectors

$$E_{\mathbf{b}} := \begin{cases} \mathbf{Z}^-, & S(\mathbf{b}) = 0, \\ \bigvee_{l=1}^{S(\mathbf{b})} E_{{}_l\mathbf{b}}, & 0 < S(\mathbf{b}) \leq n, \end{cases} \tag{4.17}$$

form a basis for the 2^n-dimensional vector space of exterior double algebra Λ_n.

We use the bold capital letter $\mathbf{Z}$ to denote the basis vector of grade 0, since it is a zero divisor with respect to the exterior product. The superscript plus or minus sign of $\mathbf{Z}$ can be omitted, if it stands for both approaches or if it is clear from the context in which approach one is working.

Notation 4.8 (Basis n-Vectors). For the n-vectors of the bases we also use the notation

$$\mathbf{I}^+ := P^+_{\mathbf{b}}, \qquad \mathbf{I}^- := E^-_{\mathbf{b}}, \qquad S(\mathbf{b}) = n. \tag{4.18}$$

The superscript plus or minus sign can be omitted, if $\mathbf{I}$ stands for both approaches or if it is clear from the context in which approach one is working. Also the n-vector $\mathbf{I}$ represents a zero divisor with respect to the exterior product.

Example. We take the unity free exterior double algebra Λ_4 of dimension $2^4 = 16$. In order to illustrate some basis vectors of this unity free exterior double algebra, let us write down two basis vectors in the plus approach

$$P_{0101} = \bigwedge_{l=1}^{2} P_{{}_l0101} = P_{0001} \wedge P_{0100} \tag{4.19}$$

$$P_{1101} = \bigwedge_{l=1}^{3} P_{{}_l1101} = P_{0001} \wedge P_{0100} \wedge P_{1000} \tag{4.20}$$

and two basis vectors in the minus approach

$$E_{0111} = \bigvee_{l=1}^{3} E_{{}_l0111} = E_{0001} \vee E_{0010} \vee E_{0100} \tag{4.21}$$

$$\mathbf{I}^{-} = E_{1111} = \bigvee_{l=1}^{4} E_{{}_l1111} = E_{0001} \vee E_{0010} \vee E_{0100} \vee E_{1000}. \tag{4.22}$$

Notation 4.9 (Generic Basis of an Exterior Algebra). To denote the basis of an exterior algebra independent of the plus or minus approach we use the expression $\{B_{\mathbf{b}}\}$, i. e. $\{B_{\mathbf{b}}^{+}\} = \{P_{\mathbf{b}}\}$ and $\{B_{\mathbf{b}}^{-}\} = \{E_{\mathbf{b}}\}$.

Proof. In order to proof Theorem 4.6 and 4.7 we need to show that the 2^n basis vectors $B_{\mathbf{b}}$ are linearly independent. According to Definition 4.2 homogeneous multi vectors of different grades are linearly independent. It remains to show, that homogeneous multi vectors $B_{\mathbf{b}}$ of the same grade $S(\mathbf{b}) = k$ are linearly independent. Trivially this is the case for $S(\mathbf{b}) = 0$ and $S(\mathbf{b}) = n$. The 1-vectors $B_{\mathbf{b}}$ are by precondition linearly independent. And for $2 \leq S(\mathbf{b}) \leq n-1$ we get with the coefficients $\lambda_{\mathbf{b}} \in \mathbb{F}$ and with the binary n-digit numbers $\mathbf{c}$

$$\mathbf{0} = \sum_{S(\mathbf{b})=k} \lambda_{\mathbf{b}} B_{\mathbf{b}} \tag{4.23}$$

$$\begin{aligned}
\Longleftrightarrow \qquad \mathbf{0} &= \left[\sum_{S(\mathbf{b})=k} \lambda_{\mathbf{b}} B_{\mathbf{b}} \right] \lozenge B_{\overline{\mathbf{c}}} \qquad \forall\ \mathbf{c} \text{ with } S(\mathbf{c}) = k && (4.24)\\
&= \sum_{S(\mathbf{b})=k} \lambda_{\mathbf{b}} \left(B_{\mathbf{b}} \lozenge B_{\overline{\mathbf{c}}} \right) \\
&= \lambda_{\mathbf{c}} \left(B_{\mathbf{c}} \lozenge B_{\overline{\mathbf{c}}} \right) \\
\Longleftrightarrow \qquad 0 &= \lambda_{\mathbf{c}}, \qquad \forall\ \mathbf{c} \text{ with } S(\mathbf{c}) = k. && (4.25)
\end{aligned}$$

□

We will now determine the multiplication tables for the bases vectors $B_{\mathbf{b}}$ with respect to the exterior products $\lozenge$. For the dimension numbers $n = 2$,

$n = 3$ and $n = 4$ the respective multiplication tables are displayed in Table 1, Table 2 and Table 3.[8]

$\diamondsuit$	$\mathbf{Z}$	B_{01}	B_{10}	$\mathbf{I}$
$\mathbf{Z}$	0	0	0	0
B_{01}	0	0	$\mathbf{I}$	0
B_{10}	0	$-\mathbf{I}$	0	0
$\mathbf{I}$	0	0	0	0

$\diamondsuit$	$\mathbf{Z}$	B_{001}	B_{010}	B_{100}	B_{011}	B_{101}	B_{110}	$\mathbf{I}$
$\mathbf{Z}$	0	0	0	0	0	0	0	0
B_{001}	0	0	B_{011}	B_{101}	0	0	$\mathbf{I}$	0
B_{010}	0	$-B_{011}$	0	B_{110}	0	$-\mathbf{I}$	0	0
B_{100}	0	$-B_{101}$	$-B_{110}$	0	$\mathbf{I}$	0	0	0
B_{011}	0	0	0	$\mathbf{I}$	0	0	0	0
B_{101}	0	0	$-\mathbf{I}$	0	0	0	0	0
B_{110}	0	$\mathbf{I}$	0	0	0	0	0	0
$\mathbf{I}$	0	0	0	0	0	0	0	0

TABLE 2. Multiplication table for the bases vectors $B_{\mathbf{b}}$ with respect to the exterior products $\diamondsuit$ in the unity free exterior double algebra Λ_3.

Theorem 4.10 (Multiplication Table for the Exterior Products in Λ_n). *Let* $\mathbf{b}, \mathbf{c}, \mathbf{d}$ *and* $\mathbf{e}$ *be binary n-digit numbers with*

$$\mathbf{d} = \mathbf{b} \text{ AND } \mathbf{c}, \tag{4.26}$$

$$\mathbf{e} = \mathbf{b} \text{ XOR } \mathbf{c}. \tag{4.27}$$

Then we have for the bases vectors $B_{\mathbf{b}}$ of the exterior double algebra Λ_n

$$B_{\mathbf{b}} \diamondsuit B_{\mathbf{c}} = \begin{cases} \alpha_{\mathbf{bc}} B_{\mathbf{e}}, & S(\mathbf{b}) \neq 0, \quad S(\mathbf{c}) \neq 0, \; S(\mathbf{d}) = 0 \\ \mathbf{0}, & S(\mathbf{b}) \neq 0, \quad S(\mathbf{c}) \neq 0, \; S(\mathbf{d}) \neq 0 \\ \mathbf{0}, & S(\mathbf{b}) = 0 \text{ or } S(\mathbf{c}) = 0 \end{cases} \tag{4.28}$$

with the coefficients

$$\alpha_{\mathbf{bc}} = (-1)^{\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m} = (-1)^{\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m}. \tag{4.29}$$

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[8] Similar considerations can be found in [Mar04] and P. LOUNESTO [Lou01, pp. 279-287]. Especially the formula for the coefficients $\alpha_{\mathbf{bc}}$ in equation (4.29) is the same as in the multiplication of Clifford elements. Compare equation (1.27) in [Mar04] and equations (8.52) and (8.53) of this article.

Proof. We determine two sequences of transpositions such that the factors, i. e. the 1-vectors of the exterior product $B_{\mathbf{b}} \Diamond B_{\mathbf{c}}$ get into the order, where equal 1-vectors are direct neighbours. To depict the order of 1-vectors in the product $B_{\mathbf{b}} \Diamond B_{\mathbf{c}}$ we use the digits of the binary numbers **b** and **c**,

$$(b_0 b_1 \dots b_{n-2} b_{n-1})(c_0 c_1 \dots c_{n-2} c_{n-1}). \tag{4.30}$$

See also the example[9] in the footnote.

First sequence of transpositions: We move the 1-vectors of $B_{\mathbf{c}}$ and shift them in terms of transpositions from the right side to the left side. Take first c_0 and shift it to the left side until it stands between b_0 and b_1, then take c_1 and shift it to the left side until it stands between b_1 and b_2 and so forth. The last transposition of this first sequence is then the shift of c_{n-2} to the left side such that it stands between b_{n-2} and b_{n-1}. The resulting order of the 1-vectors

$$(b_0 c_0 b_1 c_1 \dots b_{n-2} c_{n-2} b_{n-1} c_{n-1}) \tag{4.37}$$

was achieved by altogether

$$\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m \tag{4.38}$$

transpositions.

Second sequence of transpositions: We move the 1-vectors of $B_{\mathbf{b}}$ and shift them in terms of transpositions from the left side to the right side. Take first b_{n-1} and shift it to the right side until it stands between c_{n-2} and c_{n-1}, then take b_{n-2} and shift it to the right side until it stands between c_{n-3} and c_{n-2} and so forth. The last transposition of this second sequence is then

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[9] Let us illustrate the notation used in the equations (4.30), (4.37) and (4.38) with an example in the unity free exterior double algebra Λ_6 and by using the binary indices $\mathbf{b} = 001101$ and $\mathbf{c} = 100001$.

$$\begin{aligned} B_{\mathbf{b}} \Diamond B_{\mathbf{c}} &= B_{001101} \Diamond B_{100001} \\ &= (B_{000001} \Diamond B_{000100} \Diamond B_{001000}) \Diamond (B_{000001} \Diamond B_{100000}) \\ &\mathrel{\widehat{=}} (b_0 b_1 b_2 b_3 b_4 b_5)(c_0 c_1 c_2 c_3 c_4 c_5), \end{aligned} \tag{4.31}$$

i.e., read from the left to the right side, we have $b_0 = 1$ times the factor B_{000001}, $b_1 = 0$ times the factor B_{000010}, $b_2 = 1$ times the factor B_{000100}, $b_3 = 1$ times the factor B_{001000}, $b_4 = 0$ times the factor B_{010000}, $b_5 = 0$ times the factor B_{100000}, then $c_0 = 1$ times the factor B_{000001}, $c_1 = 0$ times the factor B_{000010}, $c_2 = 0$ times the factor B_{000100}, $c_3 = 0$ times the factor B_{001000}, $c_4 = 0$ times the factor B_{010000} and $c_5 = 1$ times the factor B_{100000}. The product above vanishes,

$$B_{\mathbf{b}} \Diamond B_{\mathbf{c}} = B_{001101} \Diamond B_{100001} = 0, \tag{4.32}$$

since we have

$$S(\mathbf{b}) = 3, \qquad S(\mathbf{c}) = 2, \tag{4.33}$$

$$\mathbf{d} = \mathbf{b} \text{ AND } \mathbf{c} = 000001, \tag{4.34}$$

$$\mathbf{e} = \mathbf{b} \text{ XOR } \mathbf{c} = 101100, \tag{4.35}$$

$$S(\mathbf{d}) = 1, \qquad S(\mathbf{e}) = 3, \tag{4.36}$$

or both factors $B_{\mathbf{b}}$ and $B_{\mathbf{c}}$ contain the 1-vector B_{000001}.

the shift of b_1 to the right side such that it stands between c_0 and c_1. The resulting order of the 1-vectors

$$(b_0 c_0 b_1 c_1 \dots b_{n-2} c_{n-2} b_{n-1} c_{n-1}). \tag{4.39}$$

was achieved by altogether

$$\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m \tag{4.40}$$

transpositions.

In both sequences the number of transpositions is the same

$$\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m = \sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m. \tag{4.41}$$

The exterior product $B_{\mathbf{b}} \Diamond B_{\mathbf{c}}$ vanishes if and only if there is at least one pair (b_l, c_l), $0 \leq l \leq n-1$, with $b_l c_l = 1$, i. e. if and only if $S(\mathbf{d}) \neq 0$. □

Corollary 4.11. *Let* $\mathbf{b}$ *be a* n*-digit binary number with* $S(\mathbf{b}) \neq 0$ *and* $S(\mathbf{b}) \neq n$. *Then we have*

$$B_{\mathbf{b}} \Diamond B_{\overline{\mathbf{b}}} = \alpha_{\mathbf{b}\overline{\mathbf{b}}} \mathbf{I}. \tag{4.42}$$

Proof. It is a consequence of Theorem 4.10 with $\mathbf{c} = \overline{\mathbf{b}}$. □

Theorem 4.12. *Let* $\mathbf{b}$ *and* $\mathbf{c}$ *be* n*-digit binary numbers. We then have*

$$\alpha_{\mathbf{bc}} \alpha_{\mathbf{cb}} = (-1)^{S(\mathbf{b})S(\mathbf{c}) - S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \tag{4.43}$$

Proof.

$$\begin{aligned} \alpha_{\mathbf{bc}} \alpha_{\mathbf{cb}} &= (-1)^{\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m + \sum_{l=1}^{n-1} c_l \sum_{m=0}^{l-1} b_m} \qquad (4.44) \\ &= (-1)^{c_0 \sum_{m=1}^{n-1} b_m + \sum_{l=1}^{n-2} c_l \left[\sum_{m=l+1}^{n-1} b_m + \sum_{m=0}^{l-1} b_m\right] + c_{n-1} \sum_{m=0}^{n-2} b_m} \\ &= (-1)^{c_0 S(\mathbf{b}) + S(\mathbf{b}) \sum_{l=1}^{n-2} c_l + c_{n-1} S(\mathbf{b}) - \left[c_0 b_0 + \sum_{l=1}^{n-2} c_l b_l + c_{n-1} b_{n-1}\right]} \\ &= (-1)^{S(\mathbf{b})S(\mathbf{c}) - S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \end{aligned}$$

□

Theorem 4.13. *For homogeneous multi vectors of grade* r *and* s *we have,*

$$X_{\overline{r}} \Diamond Y_{\overline{s}} = (-1)^{rs} \cdot Y_{\overline{s}} \Diamond X_{\overline{r}}. \tag{4.45}$$

Proof. Let $\mathbf{b}$ and $\mathbf{c}$ be n-digit binary numbers with sum of digits $S(\mathbf{b}) = r$ and $S(\mathbf{c}) = s$. Then there are coefficients $\lambda_{\mathbf{b}}$ and $\mu_{\mathbf{c}}$ such that

$$\begin{aligned} X_{\overline{r}} \Diamond Y_{\overline{s}} &= \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}}\right] \Diamond \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}}\right] \qquad (4.46) \\ &= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \cdot B_{\mathbf{b}} \Diamond B_{\mathbf{c}} \end{aligned}$$

$\Diamond$	$\mathbf{Z}$	B_{0001}	B_{0010}	B_{0100}	B_{1000}	B_{0011}	B_{0101}	B_{0110}
$\mathbf{Z}$	0	0	0	0	0	0	0	0
B_{0001}	0	0	B_{0011}	B_{0101}	B_{1001}	0	0	B_{0111}
B_{0010}	0	$-B_{0011}$	0	B_{0110}	B_{1010}	0	$-B_{0111}$	0
B_{0100}	0	$-B_{0101}$	$-B_{0110}$	0	B_{1100}	B_{0111}	0	0
B_{1000}	0	$-B_{1001}$	$-B_{1010}$	$-B_{1100}$	0	B_{1011}	B_{1101}	B_{1110}
B_{0011}	0	0	0	B_{0111}	B_{1011}	0	0	0
B_{0101}	0	0	$-B_{0111}$	0	B_{1101}	0	0	0
B_{0110}	0	B_{0111}	0	0	B_{1110}	0	0	0
B_{1001}	0	0	$-B_{1011}$	$-B_{1101}$	0	0	0	$\mathbf{I}$
B_{1010}	0	B_{1011}	0	$-B_{1110}$	0	0	$-\mathbf{I}$	0
B_{1100}	0	B_{1101}	B_{1110}	0	0	$\mathbf{I}$	0	0
B_{0111}	0	0	0	0	$\mathbf{I}$	0	0	0
B_{1011}	0	0	0	$-\mathbf{I}$	0	0	0	0
B_{1101}	0	0	$\mathbf{I}$	0	0	0	0	0
B_{1110}	0	$-\mathbf{I}$	0	0	0	0	0	0
$\mathbf{I}$	0	0	0	0	0	0	0	0

$\Diamond$	B_{1001}	B_{1010}	B_{1100}	B_{0111}	B_{1011}	B_{1101}	B_{1110}	$\mathbf{I}$
$\mathbf{Z}$	0	0	0	0	0	0	0	0
B_{0001}	0	B_{1011}	B_{1101}	0	0	0	$\mathbf{I}$	0
B_{0010}	$-B_{1011}$	0	B_{1110}	0	0	$-\mathbf{I}$	0	0
B_{0100}	$-B_{1101}$	$-B_{1110}$	0	0	$\mathbf{I}$	0	0	0
B_{1000}	0	0	0	$-\mathbf{I}$	0	0	0	0
B_{0011}	0	0	$\mathbf{I}$	0	0	0	0	0
B_{0101}	0	$-\mathbf{I}$	0	0	0	0	0	0
B_{0110}	$\mathbf{I}$	0	0	0	0	0	0	0
B_{1001}	0	0	0	0	0	0	0	0
B_{1010}	0	0	0	0	0	0	0	0
B_{1100}	0	0	0	0	0	0	0	0
B_{0111}	0	0	0	0	0	0	0	0
B_{1011}	0	0	0	0	0	0	0	0
B_{1101}	0	0	0	0	0	0	0	0
B_{1110}	0	0	0	0	0	0	0	0
$\mathbf{I}$	0	0	0	0	0	0	0	0

TABLE 3. Multiplication table for the basis vectors $B_{\mathbf{b}}$ with respect to the exterior products $\Diamond$ in the unity free exterior double algebra Λ_4.

$$\begin{aligned}
&= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot \alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}}\, B_{\mathbf{c}} \lozenge B_{\mathbf{b}} \\
&= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} (-1)^{rs}\lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot B_{\mathbf{c}} \lozenge B_{\mathbf{b}} \\
&= (-1)^{rs} \cdot \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}}\right] \lozenge \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}}\right] \\
&= (-1)^{rs} \cdot Y_{\overline{s}} \lozenge X_{\overline{r}}.
\end{aligned}$$

where we used Theorem 4.12 in the step from the third to the fourth line. □

4.1. Algebra Homomorphisms

Definition and Theorem 4.14 (Even Algebra Homomorphisms). For a fixed dimension number n and a fixed field $\mathbb{F}$ of two unity free exterior double $\mathbb{F}$-algebras Λ_n and Λ'_n let $\{P_{\mathbf{b}}\} \subset \Lambda_n$ and $\{P'_{\mathbf{b}}\} \subset \Lambda'_n$ denote two bases in the plus approach and let $\{E_{\mathbf{b}}\} \subset \Lambda_n$ and $\{E'_{\mathbf{b}}\} \subset \Lambda'_n$ denote two bases in the minus approach. Further on let the *algebra homomorphism in the plus approach*

$$\pi :\ \Lambda_n(+,\cdot,\wedge) \longrightarrow \Lambda'_n(+,\cdot,\wedge) \tag{4.47}$$

be defined by

$$\pi(P_{\mathbf{b}}) := \beta_{\mathbf{bb}} P'_{\mathbf{b}}, \qquad \beta_{\mathbf{bb}} \in \mathbb{F}, \qquad S(\mathbf{b}) = 0, \tag{4.48}$$

$$\pi(P_{\mathbf{b}}) := \sum_{S(\mathbf{c})=1} \beta_{\mathbf{bc}} P'_{\mathbf{c}}, \qquad \beta_{\mathbf{bc}} \in \mathbb{F}, \quad \forall\ S(\mathbf{b}) = 1, \tag{4.49}$$

$$\pi(A \wedge B) := \lambda \cdot \pi(A) \wedge \pi(B), \qquad \lambda \in \mathbb{F} \setminus \{0\}, \quad \forall\ A, B \in \Lambda_n, \tag{4.50}$$

and the *algebra homomorphism in the minus approach*

$$\rho :\ \Lambda_n(+,\cdot,\vee) \longrightarrow \Lambda'_n(+,\cdot,\vee) \tag{4.51}$$

be defined by

$$\rho(E_{\mathbf{b}}) := \gamma_{\mathbf{bb}} E'_{\mathbf{b}}, \qquad \gamma_{\mathbf{bb}} \in \mathbb{F}, \qquad S(\mathbf{b}) = 0, \tag{4.52}$$

$$\rho(E_{\mathbf{b}}) := \sum_{S(\mathbf{c})=1} \gamma_{\mathbf{bc}} E'_{\mathbf{c}}, \qquad \gamma_{\mathbf{bc}} \in \mathbb{F}, \quad \forall\ S(\mathbf{b}) = 1, \tag{4.53}$$

$$\rho(A \vee B) := \mu \cdot \rho(A) \vee \rho(B), \qquad \mu \in \mathbb{F} \setminus \{0\}, \quad \forall\ A, B \in \Lambda_n. \tag{4.54}$$

We can then display the two homomorphisms π and ρ by

$$\pi(P_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \beta_{\mathbf{bc}} P'_{\mathbf{c}}, \qquad 0 \le S(\mathbf{b}) = k \le n, \tag{4.55}$$

with

$$\beta_{\mathbf{bb}} \in \mathbb{F}, \qquad S(\mathbf{b}) = 0, \tag{4.56}$$

$$\beta_{\mathbf{bc}} = \lambda^{k-1} \sum_{\sigma} \operatorname{sign}\sigma \prod_{l=1}^{k} \beta_{{}_{l}\mathbf{b}_{\sigma(l)}\mathbf{c}} \in \mathbb{F}, \qquad \left\{ \begin{array}{r} S(\mathbf{b}) = S(\mathbf{c}) = k, \\ 0 < k \leq n, \end{array} \right\} \tag{4.57}$$

$$\beta_{\mathbf{bb}} = \lambda^{n-1} \det \pi \in \mathbb{F}, \qquad S(\mathbf{b}) = n. \tag{4.58}$$

and by

$$\rho(E_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \gamma_{\mathbf{bc}} E'_{\mathbf{c}}, \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{4.59}$$

with

$$\gamma_{\mathbf{bb}} \in \mathbb{F}, \qquad S(\mathbf{b}) = 0, \tag{4.60}$$

$$\gamma_{\mathbf{bc}} = \mu^{k-1} \sum_{\sigma} \operatorname{sign}\sigma \prod_{l=1}^{k} \gamma_{{}_{l}\mathbf{b}_{\sigma(l)}\mathbf{c}} \in \mathbb{F}, \qquad \left\{ \begin{array}{r} S(\mathbf{b}) = S(\mathbf{c}) = k, \\ 0 < k \leq n, \end{array} \right\} \tag{4.61}$$

$$\gamma_{\mathbf{bb}} = \mu^{n-1} \det \rho \in \mathbb{F}, \qquad S(\mathbf{b}) = n. \tag{4.62}$$

π represents an *algebra isomorphism in the plus approach* if and only if

$$\beta_{\overline{\mathbf{bb}}},\ \beta_{\mathbf{bb}} = \lambda^{n-1} \det \pi \in \mathbb{F} \setminus \{0\}, \qquad S(\mathbf{b}) = n, \tag{4.63}$$

or if and only if the set of homogeneous multi vectors $\{P''_{\mathbf{b}}\} \subset \Lambda'_n$ defined by

$$\lambda^{S(\mathbf{b})-1} P''_{\mathbf{b}} := \pi(P_{\mathbf{b}}), \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{4.64}$$

represents a basis in the plus approach of Λ'_n. ρ represents an *algebra isomorphism in the minus approach* if and only if

$$\gamma_{\overline{\mathbf{bb}}},\ \gamma_{\mathbf{bb}} = \mu^{n-1} \det \rho \in \mathbb{F} \setminus \{0\}, \qquad S(\mathbf{b}) = n, \tag{4.65}$$

or if and only if the set of homogeneous multi vectors $\{E''_{\mathbf{b}}\} \subset \Lambda'_n$ defined by

$$\mu^{S(\mathbf{b})-1} E''_{\mathbf{b}} := \rho(E_{\mathbf{b}}), \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{4.66}$$

represents a basis in the minus approach of Λ'_n.

We call π and ρ *even* algebra homomorphisms or isomorphisms.

Proof. The coefficients $\beta_{\mathbf{bb}}$ and $\gamma_{\mathbf{bb}}$ with $S(\mathbf{b}) = 0$ from equations (4.56) and (4.60) are given by definition.

Inserting $S(\mathbf{b}) = 1$ into equations (4.57) and (4.61) returns the equations (4.49) and (4.53) of the definition respectively.

For $0 < S(\mathbf{b}) = k \leq n$ and by using the permutations σ we get

$$\begin{aligned} \pi(P_{\mathbf{b}}) &= \lambda^{k-1} \bigwedge_{l=1}^{k} \pi(P_{{}_{l}\mathbf{b}}) = \lambda^{k-1} \bigwedge_{l=1}^{k} \left(\sum_{S(\mathbf{e})=1} \beta_{{}_{l}\mathbf{b}\mathbf{e}}\, P'_{\mathbf{e}} \right) \\ &= \sum_{S(\mathbf{c})=k} \left(\lambda^{k-1} \sum_{\sigma} \operatorname{sign}\sigma \prod_{l=1}^{k} \beta_{{}_{l}\mathbf{b}_{\sigma(l)}\mathbf{c}} \right) P'_{\mathbf{c}} = \sum_{S(\mathbf{c})=k} \beta_{\mathbf{bc}} P'_{\mathbf{c}} \end{aligned} \tag{4.67}$$

$$\rho(E_{\mathbf{b}}) = \mu^{k-1} \bigvee_{l=1}^{k} \rho(E_{{}_{l}\mathbf{b}}) = \mu^{k-1} \bigvee_{l=1}^{k} \left(\sum_{S(\mathbf{e})=1} \gamma_{{}_{l}\mathbf{b}\mathbf{e}}\, E'_{\mathbf{e}} \right) \tag{4.68}$$

$$= \sum_{S(\mathbf{c})=k} \left(\mu^{k-1} \sum_{\sigma} \operatorname{sign} \sigma \prod_{l=1}^{k} \gamma_{{}_l\mathbf{b}_{\sigma(l)}\mathbf{c}} \right) E'_{\mathbf{c}} = \sum_{S(\mathbf{c})=k} \gamma_{\mathbf{bc}} E'_{\mathbf{c}}$$

In the case of $S(\mathbf{b}) = n$ this is

$$\beta_{\mathbf{bb}} = \lambda^{n-1} \sum_{\sigma} \operatorname{sign} \sigma \prod_{l=1}^{n} \beta_{{}_l\mathbf{b}_{\sigma(l)}\mathbf{b}} = \lambda^{n-1} \det \pi, \tag{4.69}$$

$$\gamma_{\mathbf{bb}} = \mu^{n-1} \sum_{\sigma} \operatorname{sign} \sigma \prod_{l=1}^{n} \gamma_{{}_l\mathbf{b}_{\sigma(l)}\mathbf{b}} = \mu^{n-1} \det \rho. \tag{4.70}$$

π represents an algebra isomorphism if and only if $\beta_{\mathbf{bb}}$ with $S(\mathbf{b}) = 0$ and $\beta_{\mathbf{bb}}$ with $S(\mathbf{b}) = n$ do not vanish.

The set of homogeneous multi vectors $\{P''_{\mathbf{b}}\} \subset \Lambda'_n$ with

$$\begin{aligned} P''_{\mathbf{b}} &= \lambda^{1-S(\mathbf{b})} \pi(P_{\mathbf{b}}) \qquad\qquad 0 \leq S(\mathbf{b}) = k \leq n \\ &= \lambda^{1-S(\mathbf{b})} \lambda^{S(\mathbf{b})-1} \bigwedge_{l=1}^{S(\mathbf{b})} \pi(P_{{}_l\mathbf{b}}) \\ &= \bigwedge_{l=1}^{S(\mathbf{b})} P''_{{}_l\mathbf{b}} \end{aligned} \tag{4.71}$$

represents a basis in the plus approach of Λ'_n if and only if π is an algebra isomorphism.

ρ represents an algebra isomorphism if and only if $\gamma_{\mathbf{bb}}$ with $S(\mathbf{b}) = 0$ and $\gamma_{\mathbf{bb}}$ with $S(\mathbf{b}) = n$ do not vanish.

The set of homogeneous multi vectors $\{E''_{\mathbf{b}}\} \subset \Lambda'_n$ with

$$\begin{aligned} E''_{\mathbf{b}} &= \lambda^{1-S(\mathbf{b})} \rho(E_{\mathbf{b}}) \qquad\qquad 0 \leq S(\mathbf{b}) = k \leq n \\ &= \lambda^{1-S(\mathbf{b})} \lambda^{S(\mathbf{b})-1} \bigvee_{l=1}^{S(\mathbf{b})} \rho(E_{{}_l\mathbf{b}}) \\ &= \bigvee_{l=1}^{S(\mathbf{b})} E''_{{}_l\mathbf{b}} \end{aligned} \tag{4.72}$$

represents a basis in the minus approach of Λ'_n if and only if ρ is an algebra isomorphism. □

The even algebra homomorphisms π and ρ from Theorem (4.14) preserve the plus or minus approach. We will now define the two algebra homomorphisms $\hat{\pi}$ and $\hat{\rho}$, which are interchanging plus and minus approach.

Definition and Theorem 4.15 (Odd Algebra Homomorphisms). For a fixed dimension number n and a fixed field $\mathbb{F}$ of two unity free exterior double $\mathbb{F}$-algebras Λ_n and Λ'_n let $\{P_{\mathbf{b}}\} \subset \Lambda_n$ and $\{P'_{\mathbf{b}}\} \subset \Lambda'_n$ denote two bases in the plus approach and let $\{E_{\mathbf{b}}\} \subset \Lambda_n$ and $\{E'_{\mathbf{b}}\} \subset \Lambda'_n$ denote two bases in the minus approach. Further on let the *algebra homomorphism from the plus to the minus approach*

$$\hat{\pi}:\ \Lambda_n(+,\cdot,\wedge)\ \longrightarrow\ \Lambda'_n(+,\cdot,\vee) \tag{4.73}$$

be defined by

$$\hat{\pi}(P_{\mathbf{b}}) := \hat{\beta}_{\mathbf{bb}}E'_{\mathbf{b}}, \qquad \hat{\beta}_{\mathbf{bb}}\in\mathbb{F}, \qquad S(\mathbf{b})=0, \tag{4.74}$$

$$\hat{\pi}(P_{\mathbf{b}}) := \sum_{S(\mathbf{c})=1}\hat{\beta}_{\mathbf{bc}}E'_{\mathbf{c}}, \qquad \hat{\beta}_{\mathbf{bc}}\in\mathbb{F}, \quad \forall\ S(\mathbf{b})=1, \tag{4.75}$$

$$\hat{\pi}(A\wedge B) := \lambda\cdot\hat{\pi}(A)\vee\hat{\pi}(B), \qquad \lambda\in\mathbb{F}\setminus\{0\}, \quad \forall\ A,B\in\Lambda_n, \tag{4.76}$$

and the *algebra homomorphism from the minus to the plus approach*

$$\hat{\rho}:\ \Lambda_n(+,\cdot,\vee)\ \longrightarrow\ \Lambda'_n(+,\cdot,\wedge) \tag{4.77}$$

be defined by

$$\hat{\rho}(E_{\mathbf{b}}) := \hat{\gamma}_{\mathbf{bb}}P'_{\mathbf{b}}, \qquad \hat{\gamma}_{\mathbf{bb}}\in\mathbb{F}, \qquad S(\mathbf{b})=0, \tag{4.78}$$

$$\hat{\rho}(E_{\mathbf{b}}) := \sum_{S(\mathbf{c})=1}\hat{\gamma}_{\mathbf{bc}}P'_{\mathbf{c}}, \qquad \hat{\gamma}_{\mathbf{bc}}\in\mathbb{F}, \quad \forall\ S(\mathbf{b})=1, \tag{4.79}$$

$$\hat{\rho}(A\vee B) := \mu\cdot\hat{\rho}(A)\wedge\hat{\rho}(B), \qquad \mu\in\mathbb{F}\setminus\{0\}, \quad \forall\ A,B\in\Lambda_n. \tag{4.80}$$

We can then display the two homomorphisms $\hat{\pi}$ and $\hat{\rho}$ by

$$\hat{\pi}(P_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k}\hat{\beta}_{\mathbf{bc}}E'_{\mathbf{c}}, \qquad 0\le S(\mathbf{b})=k\le n, \tag{4.81}$$

with

$$\hat{\beta}_{\mathbf{bb}}\in\mathbb{F}, \qquad S(\mathbf{b})=0, \tag{4.82}$$

$$\hat{\beta}_{\mathbf{bc}} = \lambda^{k-1}\sum_{\sigma}\operatorname{sign}\sigma\prod_{l=1}^{k}\hat{\beta}_{{}_{l}\mathbf{b}\,{}_{\sigma(l)}\mathbf{c}}\in\mathbb{F}, \qquad \left\{\begin{array}{r} S(\mathbf{b})=S(\mathbf{c})=k, \\ 0<k\le n, \end{array}\right\} \tag{4.83}$$

$$\hat{\beta}_{\mathbf{bb}} = \lambda^{n-1}\det\hat{\pi}\in\mathbb{F}, \qquad S(\mathbf{b})=n. \tag{4.84}$$

and by

$$\hat{\rho}(E_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k}\hat{\gamma}_{\mathbf{bc}}P'_{\mathbf{c}}, \qquad 0\le S(\mathbf{b})=k\le n, \tag{4.85}$$

with

$$\hat{\gamma}_{\mathbf{bb}}\in\mathbb{F}, \qquad S(\mathbf{b})=0, \tag{4.86}$$

$$\hat{\gamma}_{\mathbf{bc}} = \mu^{k-1}\sum_{\sigma}\operatorname{sign}\sigma\prod_{l=1}^{k}\hat{\gamma}_{{}_{l}\mathbf{b}\,{}_{\sigma(l)}\mathbf{c}}\in\mathbb{F}, \qquad \left\{\begin{array}{r} S(\mathbf{b})=S(\mathbf{c})=k, \\ 0<k\le n, \end{array}\right\} \tag{4.87}$$

$$\hat{\gamma}_{\mathbf{bb}} = \mu^{n-1}\det\hat{\rho}\in\mathbb{F}, \qquad S(\mathbf{b})=n. \tag{4.88}$$

$\hat{\pi}$ represents an *algebra isomorphism from the plus to the minus approach* if and only if

$$\hat{\beta}_{\overline{\mathbf{bb}}},\ \hat{\beta}_{\mathbf{bb}} = \lambda^{n-1}\det\pi\in\mathbb{F}\setminus\{0\} \tag{4.89}$$

or if and only if the set of homogeneous multi vectors $\{E''_{\mathbf{b}}\} \subset \Lambda'_n$ defined by

$$\lambda^{S(\mathbf{b})-1} E''_{\mathbf{b}} := \hat{\pi}(P_{\mathbf{b}}), \qquad 0 \le S(\mathbf{b}) = k \le n, \tag{4.90}$$

represents a basis in the minus approach of Λ'_n. $\hat{\rho}$ represents an *algebra isomorphism from the minus to the plus approach* if and only if

$$\hat{\gamma}_{\overline{\mathbf{b}\mathbf{b}}}, \hat{\gamma}_{\mathbf{bb}} = \mu^{n-1} \det \hat{\rho} \in \mathbb{F} \setminus \{0\} \tag{4.91}$$

or if and only if the set of homogeneous multi vectors $\{P''_{\mathbf{b}}\} \subset \Lambda'_n$ defined by

$$\mu^{S(\mathbf{b})-1} P''_{\mathbf{b}} := \hat{\rho}(E_{\mathbf{b}}), \qquad 0 \le S(\mathbf{b}) = k \le n, \tag{4.92}$$

represents a basis in the plus approach of Λ'_n.

We call $\hat{\pi}$ and $\hat{\rho}$ *odd* algebra homomorphisms or isomorphisms.

Proof. This theorem is proven parallel along the steps of the proof to theorem (4.14). □

We will use here and later on the Kronecker-delta-symbol

$$\delta_{\mathbf{bc}} := \begin{cases} 1, & \mathbf{b} = \mathbf{c}, \\ 0, & \mathbf{b} \neq \mathbf{c}. \end{cases} \tag{4.93}$$

Notation 4.16 (Matrices Representing the Algebra Homomorphisms). The coefficients $\beta_{\mathbf{bc}}$ describing the algebra homomorphism π according to equation (4.55), the coefficients $\gamma_{\mathbf{bc}}$ describing the algebra homomorphism ρ according to equation (4.59), the coefficients $\hat{\beta}_{\mathbf{bc}}$ describing the algebra homomorphism $\hat{\pi}$ according to equation (4.81) and the coefficients $\hat{\gamma}_{\mathbf{bc}}$ describing the algebra homomorphism $\hat{\rho}$ according to equation (4.85) are summed up in a system of matrices respectively by

$$\underline{B}_{\overline{k}} := \left(\beta_{\mathbf{bc}}\right)_{S(\mathbf{b})=k;S(\mathbf{c})=k}, \qquad \underline{\hat{B}}_{\overline{k}} := \left(\hat{\beta}_{\mathbf{bc}}\right)_{S(\mathbf{b})=k;S(\mathbf{c})=k}, \tag{4.94}$$

$$\underline{\Gamma}_{\overline{k}} := \left(\gamma_{\mathbf{bc}}\right)_{S(\mathbf{b})=k;S(\mathbf{c})=k}, \qquad \underline{\hat{\Gamma}}_{\overline{k}} := \left(\hat{\gamma}_{\mathbf{bc}}\right)_{S(\mathbf{b})=k;S(\mathbf{c})=k}. \tag{4.95}$$

Each of the four systems can be put together into one matrix,

$$\underline{B} := \left(\beta_{\mathbf{bc}}\right)_{\mathbf{b};\mathbf{c}} \quad \text{with} \quad \beta_{\mathbf{bc}} := \begin{cases} 0, & S(\mathbf{b}) \neq S(\mathbf{c}), \\ \beta_{\mathbf{bc}}, & S(\mathbf{b}) = S(\mathbf{c}) = k, \end{cases} \tag{4.96}$$
$$= \beta_{\mathbf{bc}} \delta_{S(\mathbf{b})S(\mathbf{c})} \in \mathbb{F}$$

$$\underline{\Gamma} := \left(\gamma_{\mathbf{bc}}\right)_{\mathbf{b};\mathbf{c}} \quad \text{with} \quad \gamma_{\mathbf{bc}} := \begin{cases} 0, & S(\mathbf{b}) \neq S(\mathbf{c}), \\ \gamma_{\mathbf{bc}}, & S(\mathbf{b}) = S(\mathbf{c}) = k, \end{cases} \tag{4.97}$$
$$= \gamma_{\mathbf{bc}} \delta_{S(\mathbf{b})S(\mathbf{c})} \in \mathbb{F}$$

$$\underline{\hat{B}} := \left(\hat{\beta}_{\mathbf{bc}}\right)_{\mathbf{b};\mathbf{c}} \quad \text{with} \quad \hat{\beta}_{\mathbf{bc}} := \begin{cases} 0, & S(\mathbf{b}) \neq S(\mathbf{c}), \\ \hat{\beta}_{\mathbf{bc}}, & S(\mathbf{b}) = S(\mathbf{c}) = k, \end{cases} \tag{4.98}$$
$$= \hat{\beta}_{\mathbf{bc}} \delta_{S(\mathbf{b})S(\mathbf{c})} \in \mathbb{F}$$

$$\underline{\hat{\Gamma}} := \left(\hat{\gamma}_{\mathbf{bc}}\right)_{\mathbf{b};\mathbf{c}} \quad \text{with} \quad \hat{\gamma}_{\mathbf{bc}} := \begin{cases} 0, & S(\mathbf{b}) \neq S(\mathbf{c}), \\ \hat{\gamma}_{\mathbf{bc}}, & S(\mathbf{b}) = S(\mathbf{c}) = k, \end{cases} \tag{4.99}$$
$$= \hat{\gamma}_{\mathbf{bc}} \delta_{S(\mathbf{b})S(\mathbf{c})} \in \mathbb{F}$$

bringing together the $n+1$ equations (4.55), the $n+1$ equations (4.59), the $n+1$ equations (4.81) and the $n+1$ equations (4.85) into one equation respectively,

$$\pi(P_{\mathbf{b}}) = \sum_{\mathbf{c}} \beta_{\mathbf{bc}} P'_{\mathbf{c}}, \qquad \hat{\pi}(P_{\mathbf{b}}) = \sum_{\mathbf{c}} \hat{\beta}_{\mathbf{bc}} E'_{\mathbf{c}}, \tag{4.100}$$

$$\rho(E_{\mathbf{b}}) = \sum_{\mathbf{c}} \gamma_{\mathbf{bc}} E'_{\mathbf{c}}, \qquad \hat{\rho}(E_{\mathbf{b}}) = \sum_{\mathbf{c}} \hat{\gamma}_{\mathbf{bc}} P'_{\mathbf{c}}, \tag{4.101}$$

Notation 4.17 (Identity Matrices). In order to distinguish the notation for the n-vectors — cf. Notation 4.8 — from the notation for the identity matrices, we display the latter by

$$\mathbb{I}_n := (\delta_{ij})_{i;j}, \qquad i,j \in \{1,\ldots,n\}. \tag{4.102}$$

The subscript n is optional and can be omitted.

Notation 4.18 (Algebra Isomorphisms with an Identity Matrix). If in Theorems (4.14) and (4.15) we choose the systems of bases $\{P'_{\mathbf{b}}\}$, $\{E'_{\mathbf{b}}\} \subset \Lambda'_n$ to be identical to the systems of bases $\{P''_{\mathbf{b}}\}$, $\{E''_{\mathbf{b}}\} \subset \Lambda'_n$ respectively,

$$P'_{\mathbf{b}} = P''_{\mathbf{b}}, \qquad E'_{\mathbf{b}} = E''_{\mathbf{b}}, \qquad 0 \le S(\mathbf{b}) = k \le n, \tag{4.103}$$

and, in addition, we set the parameters λ and μ to be 1,

$$\lambda = \mu = 1, \tag{4.104}$$

then the algebra homomorphisms π, ρ, $\hat{\pi}$, $\hat{\rho}$ become mappings with an identity matrix, i.e. special isomorphisms. We denote the latter by π_0, ρ_0, $\hat{\pi}_0$, $\hat{\rho}_0$ respectively and call them *algebra isomorphisms with an identity matrix.* Using $S(\mathbf{b}) = k$ for all $\mathbf{b}$, in detail we get

$$\begin{aligned} \pi_0 :\ & \Lambda_n(+,\cdot,\wedge) && \longrightarrow && \Lambda'_n(+,\cdot,\wedge) \\ & P_{\mathbf{b}} && \longmapsto && P''_{\mathbf{b}} = \pi_0(P_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \beta_{\mathbf{bc}} P'_{\mathbf{c}} \equiv P'_{\mathbf{b}} \end{aligned} \tag{4.105}$$

$$\begin{aligned} \rho_0 :\ & \Lambda_n(+,\cdot,\vee) && \longrightarrow && \Lambda'_n(+,\cdot,\vee) \\ & E_{\mathbf{b}} && \longmapsto && E''_{\mathbf{b}} = \rho_0(E_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \gamma_{\mathbf{bc}} E'_{\mathbf{c}} \equiv E'_{\mathbf{b}} \end{aligned} \tag{4.106}$$

$$\begin{aligned} \hat{\pi}_0 :\ & \Lambda_n(+,\cdot,\wedge) && \longrightarrow && \Lambda'_n(+,\cdot,\vee) \\ & P_{\mathbf{b}} && \longmapsto && E''_{\mathbf{b}} = \hat{\pi}_0(P_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \hat{\beta}_{\mathbf{bc}} E'_{\mathbf{c}} \equiv E'_{\mathbf{b}} \end{aligned} \tag{4.107}$$

$$\begin{aligned} \hat{\rho}_0 :\ & \Lambda_n(+,\cdot,\vee) && \longrightarrow && \Lambda'_n(+,\cdot,\wedge) \\ & E_{\mathbf{b}} && \longmapsto && P''_{\mathbf{b}} = \hat{\rho}_0(E_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \hat{\gamma}_{\mathbf{bc}} P'_{\mathbf{c}} \equiv P'_{\mathbf{b}} \end{aligned} \tag{4.108}$$

with

$$\beta_{\mathbf{bc}} = \gamma_{\mathbf{bc}} = \hat{\beta}_{\mathbf{bc}} = \hat{\gamma}_{\mathbf{bc}} = \delta_{\mathbf{bc}}. \tag{4.109}$$

In case the algebra isomorphisms with identity matrix π_0, ρ_0, $\hat{\pi}_0$, $\hat{\rho}_0$ are automorphisms, i. e. $\Lambda'_n = \Lambda_n$, and all three systems of bases coincide,

$$P_{\mathbf{b}} = P'_{\mathbf{b}} = P''_{\mathbf{b}}, \qquad E_{\mathbf{b}} = E'_{\mathbf{b}} = E''_{\mathbf{b}}, \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{4.110}$$

then π_0 and ρ_0 (but neither $\hat{\pi}_0$ nor $\hat{\rho}_0$) represent identity mappings with $\pi_0(X) = X$, $\rho_0(X) = X$ for all $X \in \Lambda_n$. The identity mappings are denoted by π_{Id} and ρ_{Id}.

Theorem 4.19. *The matrices of the algebra isomorphisms π_0, ρ_0, $\hat{\pi}_0$, $\hat{\rho}_0$ from equations (4.105) to (4.108) respectively,*

$$\underline{B} = \hat{\underline{B}} = \underline{\Gamma} = \hat{\underline{\Gamma}} = \mathbb{I}_{2^n}, \tag{4.111}$$

are identity matrices of dimension 2^n with

$$\det \underline{B} = \det \hat{\underline{B}} = \det \underline{\Gamma} = \det \hat{\underline{\Gamma}} = 1. \tag{4.112}$$

Proof. See equation (4.109). □

In order to display the odd algebra homomorphisms in a more compact way, we need the *opposite* combined exterior product sign.

Definition 4.20 (Opposite Combined Exterior Product Sign). Any mathematical term which contains the opposite combined exterior product $\blacklozenge$, necessarily right after the combined exterior product $\Diamond$ was used, has to be read in the following way: If, in the first case, the combined exterior product $\Diamond$ represents the major exterior product $\wedge$, the opposite combined exterior product $\blacklozenge$ represents the minor exterior product $\vee$. And if, in the second case, the combined exterior product $\Diamond$ represents the minor exterior product $\vee$, the opposite combined exterior product $\blacklozenge$ represents the major exterior product $\wedge$.

Example. The expression

$$f\left(X \Diamond Y\right) = f\left(Y\right) \blacklozenge f\left(X\right), \qquad \forall\, X, Y \in \Lambda_n, \tag{4.113}$$

means

$$f\left(X \wedge Y\right) = f\left(Y\right) \vee f\left(X\right), \qquad \forall\, X, Y \in \Lambda_n, \tag{4.114}$$

$$f\left(X \vee Y\right) = f\left(Y\right) \wedge f\left(X\right), \qquad \forall\, X, Y \in \Lambda_n. \tag{4.115}$$

Notation 4.21 (Multiple Opposite Combined Exterior Product Sign). Analogous to the notation for multiple exterior products and the multiple combined exterior product we use for multiple opposite combined exterior products the sign

$$\mathop{\Large\blacklozenge}\limits_{l=1}^{m} X_l := X_1 \blacklozenge X_2 \blacklozenge \cdots \blacklozenge X_m. \tag{4.116}$$

For the sake of compact formulation, we introduce a notation for a generic algebra homomorphism.

Notation 4.22 (Generic Algebra Homomorphism). Let

$$\begin{array}{llll} \phi: & \Lambda_n & \longrightarrow & \Lambda'_n \\ & B_{\mathbf{b}} & \longmapsto & \phi(B_{\mathbf{b}}) = \sum_{\mathbf{c}} \kappa_{\mathbf{bc}} B'_{\mathbf{c}} \end{array} \tag{4.117}$$

denote a generic algebra homomorphism according to the Definitions 4.14 and 4.15 with

$$\phi(A \Diamond B) = \nu\, \phi(A) \Diamond \phi(B), \qquad \text{in case of } \phi \text{ is even,} \tag{4.118}$$

$$\phi(A \Diamond B) = \nu\, \phi(A) \blacklozenge \phi(B), \qquad \text{in case of } \phi \text{ is odd.} \tag{4.119}$$

for all $A, B \in \Lambda_n$, $\nu \in \mathbb{F} \setminus \{0\}$ and with the matrices

$$\underline{K} := (\kappa_{\mathbf{bc}})_{\mathbf{b};\mathbf{c}}, \qquad \underline{K_{\overline{k}}} := (\kappa_{\mathbf{bc}})_{S(\mathbf{b})=k;S(\mathbf{c})=k}, \tag{4.120}$$

which reduce to

$$\underline{K} = \underline{B}, \quad \text{in case of } \phi = \pi, \qquad \underline{K} = \underline{\hat{B}}, \quad \text{in case of } \phi = \hat{\pi}, \tag{4.121}$$

$$\underline{K} = \underline{\Gamma}, \quad \text{in case of } \phi = \rho, \qquad \underline{K} = \underline{\hat{\Gamma}}, \quad \text{in case of } \phi = \hat{\rho}. \tag{4.122}$$

Cf. also Notation 4.16.

In case ϕ represents one of the algebra isomorphisms with identity matrix π_0, ρ_0, $\hat{\pi}_0$ or $\hat{\rho}_0$ (with $\underline{K} = \mathbb{I}_{2^n}$), it is denoted by ϕ_0. In case ϕ is even and represents one of the identity mappings π_{Id} or ρ_{Id}, it is denoted by ϕ_{Id}. Cf. also Notation 4.18.

Theorem 4.23 (Inverse of an Algebra Isomorphisms). *If a generic algebra homomorphisms ϕ from Notation 4.22 represents an algebra isomorphisms, then there exists its inverse algebra isomorphisms ϕ^{-1},*

$$\phi^{-1}: \ \Lambda'_n \ \longrightarrow \ \Lambda_n \tag{4.123}$$

defined by

$$\phi^{-1}(B'_{\mathbf{b}}) := \kappa^{-1}_{\mathbf{bb}} B_{\mathbf{b}}, \qquad \kappa^{-1}_{\mathbf{bb}} \in \mathbb{F} \setminus \{0\}, \qquad S(\mathbf{b}) = 0, \tag{4.124}$$

$$\phi^{-1}(B'_{\mathbf{b}}) := \sum_{S(\mathbf{c})=1} \kappa^{-1}_{\mathbf{bc}} B_{\mathbf{c}}, \qquad \kappa^{-1}_{\mathbf{bc}} \in \mathbb{F}, \quad \forall\ S(\mathbf{b}) = 1, \tag{4.125}$$

with

$$\underline{K_{\overline{k}}}\,\underline{K_{\overline{k}}^{-1}} = \underline{K_{\overline{k}}^{-1}}\,\underline{K_{\overline{k}}} = \mathbb{I}_{\binom{n}{k}}, \qquad k \in \{0, 1\}; \tag{4.126}$$

We then get

$$\phi^{-1}\phi = \phi\phi^{-1} = \phi_{\mathrm{Id}}, \tag{4.127}$$

$$\det \phi \cdot \det(\phi^{-1}) = 1. \tag{4.128}$$

In case of ϕ being an even *algebra isomorphism with*

$$\phi(A \Diamond B) = \nu \cdot \phi(A) \Diamond \phi(B), \qquad A, B \in \Lambda_n, \tag{4.129}$$

its inverse ϕ^{-1} preserves the same exterior product with the factor ν^{-1},

$$\phi^{-1}(A' \Diamond B') = \nu^{-1} \cdot \phi^{-1}(A') \Diamond \phi^{-1}(B'), \qquad A', B' \in \Lambda'_n, \tag{4.130}$$

i. e. π *and* π^{-1} *are both algebra isomorphisms in the plus approach,* ρ *and* ρ^{-1} *are both algebra isomorphisms in the minus approach.*

In case of ϕ *being an* odd *algebra isomorphism with*

$$\phi(A \Diamond B) = \nu \cdot \phi(A) \blacklozenge \phi(B), \qquad A, B \in \Lambda_n, \tag{4.131}$$

its inverse ϕ^{-1} *preserves the opposite exterior products with the factor* ν^{-1}*,*

$$\phi^{-1}(A' \blacklozenge B') = \nu^{-1} \cdot \phi^{-1}(A') \Diamond \phi^{-1}(B'), \qquad A', B' \in \Lambda'_n, \tag{4.132}$$

i. e. $\hat{\pi}$ *and* $\hat{\pi}^{-1}$ *are algebra isomorphisms from the plus to the minus and from the minus to the plus approach respectively,* $\hat{\rho}$ *and* $\hat{\rho}^{-1}$ *are algebra isomorphisms from the minus to the plus and from the plus to the minus approach respectively.*

Proof. Let us look first at a generic even algebra isomorphism ϕ. By construction, equation (4.130) holds for some $\nu' \in \mathbb{F} \setminus \{0\}$,

$$\phi^{-1}(A' \Diamond B') = \nu' \cdot \phi^{-1}(A') \Diamond \phi^{-1}(B'), \qquad \forall\, A', B' \in \Lambda'_n. \tag{4.133}$$

We now evaluate the left side of equation (4.127) with respect to a generic basis vector with grade 2 or higher, i. e. $1 < S(\mathbf{b}) \leq n$,

$$\begin{aligned} \phi^{-1}\phi(B_{\mathbf{b}}) &= \phi^{-1}\phi\left(\mathop{\Large\Diamond}_{l=1}^{S(\mathbf{b})} B_{{}_l\mathbf{b}}\right) = \nu^{S(\mathbf{b})-1} \cdot \phi^{-1}\left(\mathop{\Large\Diamond}_{l=1}^{S(\mathbf{b})} \phi(B_{{}_l\mathbf{b}})\right) \\ &= \nu^{S(\mathbf{b})-1}(\nu')^{S(\mathbf{b})-1} \cdot \left(\mathop{\Large\Diamond}_{l=1}^{S(\mathbf{b})} \phi^{-1}\phi(B_{{}_l\mathbf{b}})\right) \\ &= \nu^{S(\mathbf{b})-1}(\nu')^{S(\mathbf{b})-1} \cdot \left(\mathop{\Large\Diamond}_{l=1}^{S(\mathbf{b})} B_{{}_l\mathbf{b}}\right) \\ &= (\nu\nu')^{S(\mathbf{b})-1} \cdot B_{\mathbf{b}}. \end{aligned} \tag{4.134}$$

$\phi^{-1}\phi$ represents the identity mapping if and only if

$$\nu' = \nu^{-1}. \tag{4.135}$$

In a similar way the evaluation of $\phi\phi^{-1}$ returns the same result. The proof for equation (4.127) is completed by evaluating $\phi^{-1}\phi$ and $\phi\phi^{-1}$ for a generic odd algebra isomorphism ϕ in an analogous way.

Equation (4.128) follows from

$$\begin{aligned} \phi^{-1}\phi(\mathbf{I}) &= \nu^{n-1} \det\phi \cdot \phi^{-1}(\mathbf{I}') \\ &= \nu^{n-1} \det\phi \left(\frac{1}{\nu}\right)^{n-1} \det(\phi^{-1}) \cdot \mathbf{I} = \det\phi \cdot \det(\phi^{-1}) \cdot \mathbf{I} = \mathbf{I}. \end{aligned} \tag{4.136}$$

As a consequence of equation (4.127), equation (4.126) holds for any grade k. ☐

In the literature about geometric algebra, the even algebra homomorphisms π and ρ defined in theorem 4.14 and the odd algebra homomorphisms $\hat{\pi}$ and $\hat{\rho}$ defined in theorem 4.15 are often called *outermorphisms.* See e. g. [HS87, LD09, LS16].

4.2. Models of the Exterior Double Algebra Λ_n

Definition 4.24 (Model of an Exterior Double Algebra Λ_n). Given an exterior double $\mathbb{F}$-algebra Λ_n, the choice of a basis in the plus approach, $\{P_{\mathbf{b}}\}$, and the choice of a basis in the minus approach, $\{E_{\mathbf{b}}\}$, determine a certain *model* of exterior double $\mathbb{F}$-algebra Λ_n.

The simplices $\{P_{\mathbf{b}}\}$ and $\{E_{\mathbf{b}}\}$ both form a basis for the same 2^n-dimensional vector space $\Lambda_n(+,\cdot)$. This is why there are two simplices transformations: One describing the basis $\{E_{\mathbf{b}}\}$ in the plus approach, the other describing the basis $\{P_{\mathbf{b}}\}$ in the minus approach.

Definition and Theorem 4.25 (Simplices Transformations ι and κ). For any model of the exterior double algebra Λ_n with a basis in the plus approach $\{P_{\mathbf{b}}\}$, which represents a first simplex, and a basis in the minus approach $\{E_{\mathbf{b}}\}$, which represents a second simplex, we denote the simplices transformations by ι and κ and define them by

$$\begin{aligned}\iota:\quad \Lambda_n(+,\cdot,\wedge) &\longrightarrow \Lambda_n(+,\cdot,\wedge) \\ P_{\mathbf{b}} &\longmapsto \iota(P_{\mathbf{b}}) := \langle\hat{\pi}_0(P_{\mathbf{b}})\rangle^+ = \langle E_{\mathbf{b}}\rangle^+ \\ &\qquad = \sum_{S(\mathbf{c})=k} \zeta_{\mathbf{b}\overline{\mathbf{c}}} P_{\overline{\mathbf{c}}}, \qquad \forall\ S(\mathbf{b})=k,\end{aligned} \tag{4.137}$$

and

$$\begin{aligned}\kappa:\quad \Lambda_n(+,\cdot,\vee) &\longrightarrow \Lambda_n(+,\cdot,\vee) \\ E_{\mathbf{b}} &\longmapsto \kappa(E_{\mathbf{b}}) := \langle\hat{\rho}_0(E_{\mathbf{b}})\rangle^- = \langle P_{\mathbf{b}}\rangle^- \\ &\qquad = \sum_{S(\mathbf{c})=k} \eta_{\mathbf{b}\overline{\mathbf{c}}} E_{\overline{\mathbf{c}}}, \qquad \forall\ S(\mathbf{b})=k,\end{aligned} \tag{4.138}$$

where $\hat{\pi}_0$ and $\hat{\rho}_0$ are the algebra isomorphisms with identity matrix defined in equations (4.107) and (4.108), here applied with $\Lambda'_n = \Lambda_n$, i. e. $\hat{\pi}_0(P_{\mathbf{b}}) = E_{\mathbf{b}}$ and $\hat{\rho}_0(E_{\mathbf{b}}) = P_{\mathbf{b}}$ for all binary numbers $\mathbf{b}$ with $0 \le S(\mathbf{b}) \le n$. Again we summarise the coefficients $\zeta_{\mathbf{b}\overline{\mathbf{c}}}$ and $\eta_{\mathbf{b}\overline{\mathbf{c}}}$ in matrices,

$$\underline{Z}_{\overline{k}} := (\zeta_{\mathbf{b}\overline{\mathbf{c}}})_{S(\mathbf{b})=k;S(\mathbf{c})=k}\,, \qquad \underline{H}_{\overline{k}} := (\eta_{\mathbf{b}\overline{\mathbf{c}}})_{S(\mathbf{b})=k;S(\mathbf{c})=k}\,, \tag{4.139}$$

and get for all $0 \le k = S(\mathbf{b}) \le n$

$$\underline{Z}_{\overline{k}}^{-1} = \underline{H}_{\overline{k}}^{T}, \qquad \underline{H}_{\overline{k}}^{-1} = \underline{Z}_{\overline{k}}^{T}. \tag{4.140}$$

ι and κ are linear one-to-one transformations satisfying $\iota\kappa^T = \kappa^T\iota = \mathrm{Id}$ and, equivalently, $\iota^T\kappa = \kappa\iota^T = \mathrm{Id}$, yet not even algebra homomorphisms as defined in Theorem 4.14.

Proof. For $0 \leq S(\mathbf{b}) = k \leq n$ we get

$$\begin{aligned} E_{\mathbf{b}} = \langle \iota(P_{\mathbf{b}}) \rangle^{-} &= \left\langle \sum_{S(\mathbf{c})=k} \zeta_{\mathbf{b}\overline{\mathbf{c}}}\, P_{\overline{\mathbf{c}}} \right\rangle^{-} = \left\langle \sum_{S(\mathbf{c})=k} \zeta_{\mathbf{b}\overline{\mathbf{c}}}\, \kappa(E_{\overline{\mathbf{c}}}) \right\rangle^{-} \\ &= \sum_{\substack{S(\mathbf{c})=k \\ S(\mathbf{d})=k}} \zeta_{\mathbf{b}\overline{\mathbf{c}}}\, \eta_{\overline{\mathbf{c}}\mathbf{d}}\, E_{\mathbf{d}} \quad \Longleftrightarrow \quad \sum_{S(\mathbf{c})=k} \zeta_{\mathbf{b}\overline{\mathbf{c}}}\, \eta_{\overline{\mathbf{c}}\mathbf{d}} = \delta_{\mathbf{b}\mathbf{d}} \end{aligned} \tag{4.141}$$

$$\begin{aligned} P_{\mathbf{b}} = \langle \kappa(E_{\mathbf{b}}) \rangle^{+} &= \left\langle \sum_{S(\mathbf{c})=k} \eta_{\mathbf{b}\overline{\mathbf{c}}}\, P_{\overline{\mathbf{c}}} \right\rangle^{+} = \left\langle \sum_{S(\mathbf{c})=k} \eta_{\mathbf{b}\overline{\mathbf{c}}}\, \iota(P_{\overline{\mathbf{c}}}) \right\rangle^{+} \\ &= \sum_{\substack{S(\mathbf{c})=k \\ S(\mathbf{d})=k}} \eta_{\mathbf{b}\overline{\mathbf{c}}}\, \zeta_{\overline{\mathbf{c}}\mathbf{d}}\, P_{\mathbf{d}} \quad \Longleftrightarrow \quad \sum_{S(\mathbf{c})=k} \eta_{\mathbf{b}\overline{\mathbf{c}}}\, \zeta_{\overline{\mathbf{c}}\mathbf{d}} = \delta_{\mathbf{b}\mathbf{d}} \end{aligned} \tag{4.142}$$

and thus

$$\underline{Z}_{\overline{k}} \underline{H}_{\overline{k}}^{T} = \underline{H}_{\overline{k}}^{T} \underline{Z}_{\overline{k}} = \mathbb{I}_{\binom{n}{k}}, \qquad \underline{H}_{\overline{k}} \underline{Z}_{\overline{k}}^{T} = \underline{Z}_{\overline{k}}^{T} \underline{H}_{\overline{k}} = \mathbb{I}_{\binom{n}{k}}. \tag{4.143}$$

ι and κ are linear by definition. They are not representing algebra homomorphisms as defined in Theorem 4.14, since they do not satisfy equation (4.48) or equation (4.52) for $n > 1$ and they do not satisfy equations (4.49) and (4.50) or equations (4.53) and (4.54) for $n > 2$. ☐

By means of the simplices transformations ι and κ of equations (4.137) and (4.138) it is now possible to compute the major exterior product $\wedge$ also in the minus approach and the minor exterior product $\vee$ also in the plus approach. Thus, the two exterior products of the unity free exterior double algebra $\Lambda_n(+, \cdot, \wedge, \vee)$ are available for any multi vectors and in any model of the algebra.

A specific model of the unity free exterior double $\mathbb{F}$-algebra is the *harmonic* one. It will significantly simplify the incidence relations in projective geometry in its coordinate-bound form later on.[10]

The harmonic model of the unity free exterior double algebra Λ_n is defined in analogy to what HANNS-JÖRG STOSS calls *Harmonisches Punkt-Geraden-System* [Sto09, Bezeichnung 2.21 on p. 29] and more general *Harmonisches Punkt-Hyperebenen-System* [Sto09, Bezeichnung 3.36 on p. 51].

Definition 4.26 (Harmonic Model of the Unity Free Exterior Double Algebra, Harmonic System of Bases). Let $\mathbf{b}$ and $\mathbf{c}$ be n-digit binary numbers. In the case of the *harmonic model of the unity free exterior double* $\mathbb{F}$*-algebra* Λ_n the simplices transformations ι and κ from Definition and Theorem 4.25 take on the form

$$\zeta_{\mathbf{b}\overline{\mathbf{c}}} := \alpha_{\mathbf{b}\overline{\mathbf{b}}} \delta_{\mathbf{b}\mathbf{c}}, \qquad \eta_{\mathbf{b}\overline{\mathbf{c}}} := (-1)^{k(n-k)} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \delta_{\mathbf{b}\mathbf{c}} = \alpha_{\overline{\mathbf{b}}\mathbf{b}} \delta_{\mathbf{b}\mathbf{c}}. \tag{4.144}$$

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[10] In the harmonic model of the unity free exterior double algebra Λ_4 a point $X = \beta_{0001} P_{0001} + \beta_{0010} P_{0010} + \beta_{0100} P_{0100} + \beta_{1000} P_{1000}$ and a plane $Y = \gamma_{0001} E_{0001} + \gamma_{0010} E_{0010} + \gamma_{0100} E_{0100} + \gamma_{1000} E_{1000}$ of projective space $\mathcal{P}_4$ will be incident, if and only if its coordinates satisfy the condition $\beta_{0001}\gamma_{0001} + \beta_{0010}\gamma_{0010} + \beta_{0100}\gamma_{0100} + \beta_{1000}\gamma_{1000} = 0$.

We denote the simplices transformations in this case by ι_0 and κ_0 and get for $0 \leq S(\mathbf{b}) = k \leq n$

$$\begin{aligned} \iota_0 : \quad & \Lambda_n(+,\cdot,\wedge) && \longrightarrow && \Lambda_n(+,\cdot,\wedge) \\ & P_{\mathbf{b}} && \longmapsto && \iota_0(P_{\mathbf{b}}) := \langle \hat{\pi}_0(P_{\mathbf{b}}) \rangle^+ = \langle E_{\mathbf{b}} \rangle^+ = \alpha_{\mathbf{b}\overline{\mathbf{b}}} P_{\overline{\mathbf{b}}} \end{aligned} \tag{4.145}$$

$$\begin{aligned} \kappa_0 : \quad & \Lambda_n(+,\cdot,\vee) && \longrightarrow && \Lambda_n(+,\cdot,\vee) \\ & E_{\mathbf{b}} && \longmapsto && \kappa_0(E_{\mathbf{b}}) := \langle \hat{\rho}_0(E_{\mathbf{b}}) \rangle^- = \langle P_{\mathbf{b}} \rangle^- = \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}}, \end{aligned} \tag{4.146}$$

with the coefficients $\alpha_{\mathbf{bc}}$ from equations (4.28) and (4.29), which translate here to

$$\alpha_{\mathbf{b}\overline{\mathbf{b}}} = (-1)^{\sum_{l=0}^{n-2} \overline{b}_l \sum_{m=l+1}^{n-1} b_m} = (-1)^{\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} \overline{b}_m}, \tag{4.147}$$

$$\alpha_{\overline{\mathbf{b}}\mathbf{b}} = (-1)^{k(n-k)} \alpha_{\mathbf{b}\overline{\mathbf{b}}}. \tag{4.148}$$

A system of bases $\{P_{\mathbf{b}}\}$, $\{E_{\mathbf{b}}\}$ in the harmonic model of the unity free exterior double $\mathbb{F}$-algebra Λ_n is called an *harmonic system of bases.*

We have to check, whether the above given definition of the matrix elements $\zeta_{\mathbf{b}\overline{\mathbf{c}}}$ and $\eta_{\mathbf{b}\overline{\mathbf{c}}}$ in equations (4.129) meet the conditions for the simplices transformations ι and κ in Definition and Theorem 4.25,

$$\begin{aligned} \sum_{S(\mathbf{c})=k} \zeta_{\mathbf{b}\overline{\mathbf{c}}} \eta_{\overline{\mathbf{c}}\mathbf{d}} &= \sum_{S(\mathbf{c})=k} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \alpha_{\mathbf{c}\overline{\mathbf{c}}} \delta_{\mathbf{bc}} \delta_{\overline{\mathbf{c}}\overline{\mathbf{d}}} \qquad S(\mathbf{b}) = k \\ &= \left(\alpha_{\mathbf{b}\overline{\mathbf{b}}}\right)^2 \delta_{\mathbf{bd}} = \delta_{\mathbf{bd}}, \end{aligned} \tag{4.149}$$

$$\begin{aligned} \sum_{S(\mathbf{c})=k} \eta_{\mathbf{b}\overline{\mathbf{c}}} \zeta_{\overline{\mathbf{c}}\mathbf{d}} &= \sum_{S(\mathbf{c})=k} \alpha_{\overline{\mathbf{b}}\mathbf{b}} \alpha_{\overline{\mathbf{c}}\mathbf{c}} \delta_{\mathbf{bc}} \delta_{\overline{\mathbf{c}}\overline{\mathbf{d}}} \\ &= \left(\alpha_{\overline{\mathbf{b}}\mathbf{b}}\right)^2 \delta_{\mathbf{bd}} = \delta_{\mathbf{bd}}. \end{aligned} \tag{4.150}$$

Example. In the harmonic model of the unity free exterior double algebra Λ_n we always have

$$\mathbf{I}^+ = \mathbf{Z}^-, \qquad \mathbf{I}^- = \mathbf{Z}^+. \tag{4.151}$$

Theorem 4.27. *In the harmonic model of the unity free exterior double algebra we have for the k-vectors*

$$X^+_{\overline{k}} = \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}}, \qquad Y^-_{\overline{k}} = \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}} E_{\mathbf{b}}, \qquad 0 < k < n, \tag{4.152}$$

the two (equivalent) equations

$$X^+_{\overline{k}} \wedge Y^-_{\overline{k}} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] \mathbf{I}^+, \quad X^+_{\overline{k}} \vee Y^-_{\overline{k}} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] \mathbf{I}^-. \tag{4.153}$$

Proof.

$$X^+_{\overline{k}} \wedge Y^-_{\overline{k}} = \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{d}) = k}} \mu_{\mathbf{b}} \nu_{\mathbf{d}} P_{\mathbf{b}} \wedge E_{\mathbf{d}} = \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{d}) = k}} \alpha_{\mathbf{d}\overline{\mathbf{d}}} \mu_{\mathbf{b}} \nu_{\mathbf{d}} P_{\mathbf{b}} \wedge P_{\overline{\mathbf{d}}} \tag{4.154}$$

$$
\begin{aligned}
&= \left[\sum_{S(\mathbf{b})=k} (\alpha_{\mathbf{b}\overline{\mathbf{b}}})^2 \mu_{\mathbf{b}} \nu_{\mathbf{b}}\right] \mathbf{I}^+ = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}}\right] \mathbf{I}^+ \\
X_{\overline{k}}^+ \vee Y_{\overline{k}}^- &= \sum_{\substack{S(\mathbf{b})=k \\ S(\mathbf{d})=k}} \mu_{\mathbf{b}} \nu_{\mathbf{d}} P_{\mathbf{b}} \vee E_{\mathbf{d}} = \sum_{\substack{S(\mathbf{b})=k \\ S(\mathbf{d})=k}} \alpha_{\overline{\mathbf{b}}\mathbf{b}} \mu_{\mathbf{b}} \nu_{\mathbf{d}} E_{\overline{\mathbf{b}}} \vee E_{\mathbf{d}} \qquad (4.155) \\
&= \left[\sum_{S(\mathbf{d})=k} (\alpha_{\overline{\mathbf{d}}\mathbf{d}})^2 \mu_{\mathbf{d}} \nu_{\mathbf{d}}\right] \mathbf{I}^- = \left[\sum_{S(\mathbf{d})=k} \mu_{\mathbf{d}} \nu_{\mathbf{d}}\right] \mathbf{I}^-
\end{aligned}
$$

□

Any model of the exterior double algebra with its simplices transformations ι and κ can be decomposed into the simplices transformations ι_0 and κ_0 respectively and an even algebra automorphism.

Definition 4.28. Let the exterior double algebra Λ_n be endowed with a generic model expressed in terms of the simplices transformations ι and κ according to Definition and Theorem 4.25. Then ι and κ can be decomposed into the transformations ι_0 and κ_0 of the harmonic model respectively plus an even algebra automorphism,

$$\iota = \iota_0 \pi_2 = \pi_1 \iota_0, \qquad \kappa = \kappa_0 \rho_2 = \rho_1 \kappa_0. \qquad (4.156)$$

Equations (4.156) define the even algebra automorphisms

$$
\begin{aligned}
\pi_i &: \Lambda(+,\cdot,\wedge) \longrightarrow \Lambda(+,\cdot,\wedge), \quad \pi_1 := \iota\kappa_0^T, \quad \pi_2 := \kappa_0^T \iota, \quad i \in \{1,2\}, \qquad (4.157) \\
\rho_i &: \Lambda(+,\cdot,\vee) \longrightarrow \Lambda(+,\cdot,\vee), \quad \rho_1 := \kappa\iota_0^T, \quad \rho_2 := \iota_0^T \kappa.
\end{aligned}
$$

In this article, from now on, we will work in the harmonic model only.

4.3. Dual Spaces in Exterior Double Algebra Λ_n

Let V be any finite dimensional vector space over a field $\mathbb{F}$. It is standard in linear algebra to define the *dual vector space* V^* as the set of all linear mappings (linear functionals) $\phi : V \longmapsto \mathbb{F}$. The dual space V^* again is a vector space over the field $\mathbb{F}$ with the same dimension $\dim V$. The *natural pairing*

$$
\begin{aligned}
B_D : \quad V \times V^* &\longrightarrow \mathbb{F} \qquad (4.158) \\
(X, \phi) &\longmapsto B_D(X, \phi) := \phi(X)
\end{aligned}
$$

defines the non degenerate bilinear form B_D. The double dual space V^{**} naturally is isomorph to V.

In the context of the exterior double algebra Λ_n the natural pairing B_D can be defined over the Cartesian product of the plus and the minus approach. In order to do so, we need the one-to-one correspondence between linear functionals and multi vectors.

Theorem 4.29. *Let $V \subset \Lambda_n^+$ be a sub vector space of the exterior double algebra in the plus approach and let $\phi \in V^*$ be a functional of its dual space. Then there is one and only one multi vector in the minus approach*

$$Y = \sum_{\mathbf{c}} \nu_{\mathbf{c}} E_{\mathbf{c}} \in \Lambda_n^-, \tag{4.159}$$

such that

$$\phi(X) = \sum_{\mathbf{b}} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \in \mathbb{F}, \qquad \forall\ X = \sum_{\mathbf{b}} \mu_{\mathbf{b}} P_{\mathbf{b}} \in V \subset \Lambda_n^+. \tag{4.160}$$

In the case of $V \subset \Lambda_n^{k+}$ representing a sub vector space of the k-vectors in the plus approach, $Y = \langle Y \rangle_k^-$ is a k-vector in the minus approach and the functional is

$$\phi(X)\mathbf{Z}^+ = \begin{cases} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \mathbf{Z}^+ & X = \langle X \rangle_0,\ S(\mathbf{b}) = 0, \\ X \vee Y = \sum\limits_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \mathbf{Z}^+ & X = \langle X \rangle_k,\ 0 < k < n, \\ \mu_{\mathbf{b}} \nu_{\mathbf{b}} \mathbf{Z}^+ & X = \langle X \rangle_n,\ S(\mathbf{b}) = n. \end{cases} \tag{4.161}$$

Let $V \subset \Lambda_n^-$ be a sub vector space of the exterior double algebra in the minus approach and let $\psi \in V^*$ be a functional of its dual space. Then there is one and only one multi vector in the plus approach

$$X = \sum_{\mathbf{c}} \mu_{\mathbf{c}} P_{\mathbf{c}} \in \Lambda_n^+, \tag{4.162}$$

such that

$$\psi(Y) = \sum_{\mathbf{b}} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \in \mathbb{F}, \qquad \forall\ Y = \sum_{\mathbf{b}} \nu_{\mathbf{b}} E_{\mathbf{b}} \in V \subset \Lambda_n^-. \tag{4.163}$$

In the case of $V \subset \Lambda_n^{k-}$ representing a sub vector space of the k-vectors in the minus approach, $X = \langle X \rangle_k^+$ is a k-vector in the plus approach and the functional is

$$\psi(Y)\mathbf{Z}^- = \begin{cases} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \mathbf{Z}^- & Y = \langle Y \rangle_0,\ S(\mathbf{b}) = 0, \\ X \wedge Y = \sum\limits_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \mathbf{Z}^- & Y = \langle Y \rangle_k,\ 0 < k < n, \\ \mu_{\mathbf{b}} \nu_{\mathbf{b}} \mathbf{Z}^- & Y = \langle Y \rangle_n,\ S(\mathbf{b}) = n. \end{cases} \tag{4.164}$$

Proof. If we choose $X = P_{\mathbf{c}}$ in equation (4.160) and $Y = E_{\mathbf{c}}$ in equation (4.163), we get

$$\phi(P_{\mathbf{c}}) = \nu_{\mathbf{c}}, \qquad \psi(E_{\mathbf{c}}) = \mu_{\mathbf{c}}. \tag{4.165}$$

This is why to each functional ϕ or ψ corresponds one and only one multi vector Y in the minus approach or one multi vector X in the plus approach respectively. Equations (4.161) follow from equations (4.160) and Theorem (4.27). Equations (4.164) follow from equations (4.163) and Theorem (4.27).

The linear functionals $\phi(X)$ in equation (4.161) and $\psi(Y)$ in equation (4.164) can be computed in terms of the minor or major exterior product respectively, if and only if $\langle X \rangle_0 = 0$ and $\langle X \rangle_n = 0$ or if and only if $\langle Y \rangle_0 = 0$ and $\langle Y \rangle_n = 0$ respectively. □

Definition and Theorem 4.30 (Dual Spaces in Exterior Double Algebra Λ_n). Let $A \subset \Lambda_n^+$ represent a sub vector space of the exterior double algebra in the plus approach, let $B \subset \Lambda_n^-$ represent a sub vector space of the exterior double algebra in the minus approach Λ_n^-, let X be any multi vector of A and Y any multi vector of B,

$$X = \sum_{\mathbf{b}} \mu_{\mathbf{b}} P_{\mathbf{b}} \in A \subset \Lambda_n^+, \qquad Y = \sum_{\mathbf{b}} \nu_{\mathbf{b}} E_{\mathbf{b}} \in B \subset \Lambda_n^-. \tag{4.166}$$

The bilinear form over the Cartesian product of the plus and the minus approach defined by

$$\begin{aligned} B_D : \quad & A \times B && \longrightarrow \quad \mathbb{F} \\ (X,Y) = & \left(\sum_{\mathbf{b}} \mu_{\mathbf{b}} P_{\mathbf{b}}, \sum_{\mathbf{b}} \nu_{\mathbf{b}} E_{\mathbf{b}} \right) && \longmapsto \quad B_D(X,Y) := \sum_{\mathbf{b}} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \end{aligned} \tag{4.167}$$

for all $X \in A$ and for all $Y \in B$ represents a natural pairing, if and only if B_D is non degenerate. In the latter case the sub vector spaces $A \subset \Lambda_n^+$ and $B \subset \Lambda_n^-$ are dual to each other, i. e.

$$B = A^*, \qquad A = B^*, \tag{4.168}$$

and we have

$$A = A^{**}, \qquad B = B^{**}. \tag{4.169}$$

Some special cases of dual and double dual (sub) vector spaces are

$$(\Lambda_n^+)^* = \Lambda_n^-, \quad (\Lambda_n^-)^* = \Lambda_n^+, \quad (\Lambda_n^{k+})^* = \Lambda_n^{k-}, \quad (\Lambda_n^{k-})^* = \Lambda_n^{k+}, \tag{4.170}$$

$$(\Lambda_n^+)^{**} = \Lambda_n^+, \quad (\Lambda_n^-)^{**} = \Lambda_n^-, \quad (\Lambda_n^{k+})^{**} = \Lambda_n^{k+}, \quad (\Lambda_n^{k-})^{**} = \Lambda_n^{k-}. \tag{4.171}$$

Proof. The bilinear form defined in equation (4.167) in general is non degenerate or degenerate. In case of a non degenerate bilinear form B_D, there is according to Theorem (4.29) for every $Y \in B$ a non degenerate $\phi(X) = B_D(X,Y)$ and for every $X \in B$ a non degenerate $\psi(Y) = B_D(X,Y)$. This is why equations (4.168) are true.

Equations (4.169) follow from equations (4.168),

$$A = B^* = (A^*)^* = A^{**}, \qquad B = A^* = (B^*)^* = B^{**}. \tag{4.172}$$

The natural pairings B_D corresponding to the dual (sub) vector spaces of equations (4.170) are displayed in equations (4.160), (4.163), (4.161) and (4.164) respectively.

Equations (4.171) follow from equations (4.170). ☐

4.4. Dual Algebra Homomorphisms

Following the standard procedures of linear algebra, we are going to define dual or transpose algebra homomorphisms. Given the two $\mathbb{F}$-vector spaces V and W, as well as the linear map $f : V \longrightarrow W$, the dual or transpose map f^* is defined by $f^* : W^* \longrightarrow V^*$ with $B_D(X, \phi) = B_D(X, f^*(\psi)) = B_D(f(X), \psi) = B_D(Y, \psi)$ for all $X \in V$, $\phi = f^*(\psi) \in V^*$, $Y = f(X) \in W$ and $\psi \in W^*$.

Definition and Theorem 4.31 (Dual Algebra Homomorphisms). Given are the algebra homomorphisms π, ρ, $\hat{\pi}$ and $\hat{\rho}$ as defined in Definition and Theorem 4.14 and 4.15. In order to define the *dual* or *transpose* algebra homomorphisms π^*, ρ^*, $\hat{\pi}^*$ and $\hat{\rho}^*$ respectively, we will use $0 \leq S(\mathbf{b}) = k \leq n$.

The dual algebra homomorphism

$$\begin{aligned} \pi^* : \quad & \Lambda_n(+,\cdot,\vee) & \longleftarrow \quad & \Lambda'_n(+,\cdot,\vee) \\ & \pi^*(Y') = \sum_{\mathbf{b}} \nu'_{\mathbf{b}} \cdot \pi^*(E'_{\mathbf{b}}) & \longleftarrow\!\shortmid \quad & Y' = \sum_{\mathbf{b}} \nu'_{\mathbf{b}} E'_{\mathbf{b}} \\ & \pi^*(E'_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \beta^*_{\mathbf{bc}} E_{\mathbf{c}} & \longleftarrow\!\shortmid \quad & E'_{\mathbf{b}} \end{aligned} \tag{4.173}$$

with respect to the algebra homomorphism π is defined by

$$B_D(X, \pi^*(Y')) = B_D(\pi(X), Y'), \quad X = \sum_{\mathbf{b}} \mu_{\mathbf{b}} P_{\mathbf{b}} \in \Lambda_n(+,\cdot,\wedge). \tag{4.174}$$

The dual algebra homomorphism

$$\begin{aligned} \rho^* : \quad & \Lambda_n(+,\cdot,\wedge) & \longleftarrow \quad & \Lambda'_n(+,\cdot,\wedge) \\ & \rho^*(X') = \sum_{\mathbf{b}} \mu'_{\mathbf{b}} \cdot \rho^*(P'_{\mathbf{b}}) & \longleftarrow\!\shortmid \quad & X' = \sum_{\mathbf{b}} \mu'_{\mathbf{b}} P'_{\mathbf{b}} \\ & \rho^*(P'_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \gamma^*_{\mathbf{bc}} P_{\mathbf{c}} & \longleftarrow\!\shortmid \quad & P'_{\mathbf{b}} \end{aligned} \tag{4.175}$$

with respect to the algebra homomorphism ρ is defined by

$$B_D(\rho^*(X'), Y) = B_D(X', \rho(Y)), \quad Y = \sum_{\mathbf{b}} \nu_{\mathbf{b}} E_{\mathbf{b}} \in \Lambda_n(+,\cdot,\vee). \tag{4.176}$$

The dual algebra homomorphism

$$\begin{aligned} \hat{\pi}^* : \quad & \Lambda_n(+,\cdot,\vee) & \longleftarrow \quad & \Lambda'_n(+,\cdot,\wedge) \\ & \hat{\pi}^*(X') = \sum_{\mathbf{b}} \mu'_{\mathbf{b}} \cdot \hat{\pi}^*(P'_{\mathbf{b}}) & \longleftarrow\!\shortmid \quad & X' = \sum_{\mathbf{b}} \mu'_{\mathbf{b}} P'_{\mathbf{b}} \\ & \hat{\pi}^*(P'_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \hat{\beta}^*_{\mathbf{bc}} E_{\mathbf{c}} & \longleftarrow\!\shortmid \quad & P'_{\mathbf{b}} \end{aligned} \tag{4.177}$$

with respect to the algebra homomorphism $\hat{\pi}$ is defined by

$$B_D(X, \hat{\pi}^*(X')) = B_D(X', \hat{\pi}(X)), \quad X = \sum_{\mathbf{b}} \mu_{\mathbf{b}} P_{\mathbf{b}} \in \Lambda_n(+,\cdot,\wedge). \tag{4.178}$$

And the dual algebra homomorphism

$$\begin{aligned} \hat{\rho}^* : \quad & \Lambda_n(+,\cdot,\wedge) & \longleftarrow \quad & \Lambda'_n(+,\cdot,\vee) \\ & \hat{\rho}^*(Y') = \sum_{\mathbf{b}} \nu'_{\mathbf{b}} \cdot \hat{\rho}^*(E'_{\mathbf{b}}) & \longleftarrow\!\shortmid \quad & Y' = \sum_{\mathbf{b}} \nu'_{\mathbf{b}} E'_{\mathbf{b}} \\ & \hat{\rho}^*(E'_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \hat{\gamma}^*_{\mathbf{bc}} P_{\mathbf{c}} & \longleftarrow\!\shortmid \quad & E'_{\mathbf{b}} \end{aligned} \tag{4.179}$$

with respect to the algebra homomorphism $\hat{\rho}$ is defined by

$$B_D(\hat{\rho}^*(Y'),Y) = B_D(\hat{\rho}(Y),Y'), \qquad Y = \sum_{\mathbf{b}} \nu_{\mathbf{b}} E_{\mathbf{b}} \in \Lambda_n(+,\cdot,\vee). \tag{4.180}$$

The matrices representing the dual algebra homomorphisms π^*, ρ^*, $\hat{\pi}^*$ and $\hat{\rho}^*$ are transposed with respect to their corresponding algebra homomorphisms π, ρ, $\hat{\pi}$ and $\hat{\rho}$ respectively,

$$\underline{B}^*_{\underline{k}} = \underline{B}^T_{\underline{k}}, \qquad \underline{\Gamma}^*_{\underline{k}} = \underline{\Gamma}^T_{\underline{k}}, \qquad \hat{\underline{B}}^*_{\underline{k}} = \hat{\underline{B}}^T_{\underline{k}}, \qquad \hat{\underline{\Gamma}}^*_{\underline{k}} = \hat{\underline{\Gamma}}^T_{\underline{k}}. \tag{4.181}$$

π^* represents an algebra homomorphism in the minus approach, ρ^* represents an algebra homomorphism in the plus approach, $\hat{\pi}^*$ represents an algebra homomorphism from the plus to the minus approach and $\hat{\rho}^*$ represents an algebra homomorphism from the minus to the plus approach,

$$\pi^*(X' \vee Y') = \lambda \cdot \pi^*(X') \vee \pi^*(Y') \qquad \forall\, X', Y' \in \Lambda'_n(+,\cdot,\vee), \tag{4.182}$$

$$\rho^*(X' \wedge Y') = \mu \cdot \rho^*(X') \wedge \rho^*(Y') \qquad \forall\, X', Y' \in \Lambda'_n(+,\cdot,\wedge), \tag{4.183}$$

$$\hat{\pi}^*(X' \wedge Y') = \lambda \cdot \hat{\pi}^*(X') \vee \hat{\pi}^*(Y') \qquad \forall\, X', Y' \in \Lambda'_n(+,\cdot,\wedge), \tag{4.184}$$

$$\hat{\rho}^*(X' \vee Y') = \mu \cdot \hat{\rho}^*(X') \wedge \hat{\rho}^*(Y') \qquad \forall\, X', Y' \in \Lambda'_n(+,\cdot,\vee), \tag{4.185}$$

with

$$\det(\pi^*) = \det \pi, \qquad \det(\rho^*) = \det \rho, \tag{4.186}$$

$$\det(\hat{\pi}^*) = \det \hat{\pi}, \qquad \det(\hat{\rho}^*) = \det \hat{\rho}. \tag{4.187}$$

Proof. Inserting the different bases elements into equations (4.174), (4.176), (4.178), (4.180) and using the natural pairing B_D defined in Definition and Theorem (4.30) as well as $0 \leq S(\mathbf{b}) = S(\mathbf{c}) = k \leq n$, we get

$$\begin{aligned} \beta^*_{\mathbf{cb}} &= \sum_{S(\mathbf{d})=k} \beta^*_{\mathbf{cd}} B_D(P_{\mathbf{b}}, E_{\mathbf{d}}) = B_D(P_{\mathbf{b}}, \pi^*(E'_{\mathbf{c}})) \\ &= B_D(\pi(P_{\mathbf{b}}), E'_{\mathbf{c}}) = \sum_{S(\mathbf{d})=k} \beta_{\mathbf{bd}} B_D(P'_{\mathbf{d}}, E'_{\mathbf{c}}) = \beta_{\mathbf{bc}}, \end{aligned} \tag{4.188}$$

$$\begin{aligned} \gamma^*_{\mathbf{bc}} &= \sum_{S(\mathbf{d})=k} \gamma^*_{\mathbf{bd}} B_D(P_{\mathbf{d}}, E_{\mathbf{c}}) = B_D(\rho^*(P'_{\mathbf{b}}), E_{\mathbf{c}}) \\ &= B_D(P'_{\mathbf{b}}, \rho(E_{\mathbf{c}})) = \sum_{S(\mathbf{d})=k} \gamma_{\mathbf{cd}} B_D(P'_{\mathbf{b}}, E'_{\mathbf{d}}) = \gamma_{\mathbf{cb}}, \end{aligned} \tag{4.189}$$

$$\begin{aligned} \hat{\beta}^*_{\mathbf{cb}} &= \sum_{S(\mathbf{d})=k} \hat{\beta}^*_{\mathbf{cd}} B_D(P_{\mathbf{b}}, E_{\mathbf{d}}) = B_D(P_{\mathbf{b}}, \hat{\pi}^*(P'_{\mathbf{c}})) \\ &= B_D(P'_{\mathbf{c}}, \hat{\pi}(P_{\mathbf{b}})) = \sum_{S(\mathbf{d})=k} \hat{\beta}_{\mathbf{bd}} B_D(P'_{\mathbf{c}}, E'_{\mathbf{d}}) = \hat{\beta}_{\mathbf{bc}}, \end{aligned} \tag{4.190}$$

$$\begin{aligned} \hat{\gamma}^*_{\mathbf{bc}} &= \sum_{S(\mathbf{d})=k} \hat{\gamma}^*_{\mathbf{bd}} B_D(P_{\mathbf{d}}, E_{\mathbf{c}}) = B_D(\hat{\rho}^*(E'_{\mathbf{b}}), E_{\mathbf{c}}) \\ &= B_D(\hat{\rho}(E_{\mathbf{c}}), E'_{\mathbf{b}}) = \sum_{S(\mathbf{d})=k} \hat{\gamma}_{\mathbf{cd}} B_D(P'_{\mathbf{d}}, E'_{\mathbf{b}}) = \hat{\gamma}_{\mathbf{cb}}, \end{aligned} \tag{4.191}$$

i. e., equations (4.181) hold.

In case of the dual algebra homomorphism π^*, we have with equation (4.57) from Definition and Theorem 4.14 and with $m = \sigma^{-1}(l)$

$$\begin{aligned}\beta^*_{\mathbf{bc}} = \beta_{\mathbf{cb}} &= \lambda^{k-1} \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) \prod_{l=1}^{k} \beta_{[{}_l\mathbf{c}][{}_{\sigma^{-1}(l)}\mathbf{b}]} \\ &= \lambda^{k-1} \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) \prod_{l=1}^{k} \beta^*_{[{}_{\sigma^{-1}(l)}\mathbf{b}][{}_l\mathbf{c}]} = \lambda^{k-1} \sum_{\sigma} \operatorname{sign}\sigma \prod_{m=1}^{k} \beta^*_{[{}_m\mathbf{b}][{}_{\sigma(m)}\mathbf{c}]}\end{aligned} \tag{4.192}$$

for all $\mathbf{b}$, $\mathbf{c}$ with $0 < S(\mathbf{b}) = S(\mathbf{c}) = k \leq n$. For $S(\mathbf{b}) = S(\mathbf{c}) = 0$ there is just the number $\beta^*_{\mathbf{bb}} = \beta_{\mathbf{bb}}$. We can now show the dual algebra homomorphism π^* is preserving the minor exterior product according to equation (4.182). Because of the linearity of π^*, it is enough to show equation (4.182) for any bases elements $X' = E'_{\mathbf{b}}$ and $Y' = E'_{\mathbf{c}}$ with $S(\mathbf{b}) = k$ and $S(\mathbf{c}) = l$. In case of $S(\mathbf{b}) \neq 0$, $S(\mathbf{c}) \neq 0$ and $S(\mathbf{d}) = S(\mathbf{b}\ \text{AND}\ \mathbf{c}) = 0$ we get

$$\begin{aligned}\lambda \cdot \pi^*(E'_{\mathbf{b}}) \vee \pi^*(E'_{\mathbf{c}}) &= \lambda \cdot \left(\sum_{S(\mathbf{f})=k} \beta^*_{\mathbf{bf}} E_{\mathbf{f}}\right) \vee \left(\sum_{S(\mathbf{g})=l} \beta^*_{\mathbf{cg}} E_{\mathbf{g}}\right) \\ &= \lambda\lambda^{k-1}\lambda^{l-1} \cdot \left(\bigvee_{i=1}^{k} \pi^*(E'_{{}_i\mathbf{b}})\right) \vee \left(\bigvee_{j=1}^{l} \pi^*(E'_{{}_j\mathbf{c}})\right) \\ &= \alpha_{\mathbf{bc}}\, \lambda^{(k+l)-1} \cdot \bigvee_{i=1}^{k+l} \pi^*(E'_{{}_i\mathbf{e}}) = \alpha_{\mathbf{bc}} \left(\sum_{S(\mathbf{h})=k+l} \beta^*_{\mathbf{eh}} E_{\mathbf{h}}\right) \\ &= \alpha_{\mathbf{bc}} \cdot \pi^*(E'_{\mathbf{e}}) = \pi^*(\alpha_{\mathbf{bc}} \cdot E'_{\mathbf{e}}) = \pi^*(E'_{\mathbf{b}} \vee E'_{\mathbf{c}}),\end{aligned} \tag{4.193}$$

where we were using equations (4.192) and (4.173), the arguments of equation (4.68), the associativity of the minor exterior product, Theorem (4.10) and the binary number $\mathbf{e} = \mathbf{b}\ \text{XOR}\ \mathbf{c}$. In case of $S(\mathbf{b}) \neq 0$, $S(\mathbf{c}) \neq 0$ and $S(\mathbf{d}) \neq 0$ we use equation (4.193) to see both sides of equation (4.182) vanishing. In case of $S(\mathbf{b}) = 0$ or $S(\mathbf{c}) = 0$ both sides of equation (4.182) vanish trivially.

This is why π^* represents an algebra homomorphism in the minus approach and preserves the minor exterior product according to equation (4.182).

In a similar way it is possible to prove, ρ^* represents an algebra homomorphism in the plus approach and preserves the major exterior product according to equation (4.183).

In case of the dual algebra homomorphism $\hat{\pi}^*$, we have with equation (4.83) from Definition and Theorem 4.15 and with $m = \sigma^{-1}(l)$

$$\hat{\beta}^*_{\mathbf{bc}} = \hat{\beta}_{\mathbf{cb}} = \lambda^{k-1} \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) \prod_{l=1}^{k} \hat{\beta}_{[{}_l\mathbf{c}][{}_{\sigma^{-1}(l)}\mathbf{b}]} \tag{4.194}$$

$$= \lambda^{k-1} \sum_{\sigma^{-1}} \operatorname{sign}(\sigma^{-1}) \prod_{l=1}^{k} \hat{\beta}^*_{[{}_{\sigma^{-1}(l)}\mathbf{b}][{}_l\mathbf{c}]} = \lambda^{k-1} \sum_{\sigma} \operatorname{sign}\sigma \prod_{m=1}^{k} \hat{\beta}^*_{[{}_m\mathbf{b}][{}_{\sigma(m)}\mathbf{c}]}$$

for all $\mathbf{b}$, $\mathbf{c}$ with $0 < S(\mathbf{b}) = S(\mathbf{c}) = k \leq n$. For $S(\mathbf{b}) = S(\mathbf{c}) = 0$ there is just the number $\hat{\beta}^*_{\mathbf{bb}} = \hat{\beta}_{\mathbf{bb}}$. We can now show the dual algebra homomorphism $\hat{\pi}^*$ is preserving the exterior products according to equation (4.184). Because of the linearity of $\hat{\pi}^*$, it is enough to show equation (4.184) for any bases elements $X' = P'_{\mathbf{b}}$ and $Y' = P'_{\mathbf{c}}$ with $S(\mathbf{b}) = k$ and $S(\mathbf{c}) = l$. In case of $S(\mathbf{b}) \neq 0$, $S(\mathbf{c}) \neq 0$ and $S(\mathbf{d}) = S(\mathbf{b} \text{ AND } \mathbf{c}) = 0$ we get

$$\begin{aligned} \lambda \cdot \hat{\pi}^*(P'_{\mathbf{b}}) \vee \hat{\pi}^*(P'_{\mathbf{c}}) &= \lambda \cdot \left(\sum_{S(\mathbf{f})=k} \hat{\beta}^*_{\mathbf{bf}} E_{\mathbf{f}} \right) \vee \left(\sum_{S(\mathbf{g})=l} \hat{\beta}^*_{\mathbf{cg}} E_{\mathbf{g}} \right) \qquad (4.195) \\ &= \lambda \lambda^{k-1} \lambda^{l-1} \cdot \left(\bigvee_{i=1}^{k} \hat{\pi}^*({P'}_{{}_i\mathbf{b}}) \right) \vee \left(\bigvee_{j=1}^{l} \hat{\pi}^*({P'}_{{}_j\mathbf{c}}) \right) \\ &= \alpha_{\mathbf{bc}}\, \lambda^{(k+l)-1} \cdot \bigvee_{i=1}^{k+l} \hat{\pi}^*({P'}_{{}_i\mathbf{e}}) = \alpha_{\mathbf{bc}} \left(\sum_{S(\mathbf{h})=k+l} \hat{\beta}^*_{\mathbf{eh}} E_{\mathbf{h}} \right) \\ &= \alpha_{\mathbf{bc}} \cdot \hat{\pi}^*(P'_{\mathbf{e}}) = \hat{\pi}^*(\alpha_{\mathbf{bc}} \cdot P'_{\mathbf{e}}) = \hat{\pi}^*(P'_{\mathbf{b}} \wedge P'_{\mathbf{c}}), \end{aligned}$$

where we were using equations (4.194) and (4.177), the arguments of equation (4.68), the associativity of the exterior products, Theorem (4.10) and the binary number $\mathbf{e} = \mathbf{b}$ XOR $\mathbf{c}$. In case of $S(\mathbf{b}) \neq 0$, $S(\mathbf{c}) \neq 0$ and $S(\mathbf{d}) \neq 0$ we use equation (4.195) to see both sides of equation (4.184) vanishing. In case of $S(\mathbf{b}) = 0$ or $S(\mathbf{c}) = 0$ both sides of equation (4.184) vanish trivially.

This is why $\hat{\pi}^*$ represents an algebra homomorphism from the plus to the minus approach and preserves the exterior products according to equation (4.184).

In a similar way it is possible to prove, $\hat{\rho}^*$ represents an algebra homomorphism from the minus to the plus approach and perserves the exterior products according to equation (4.185).

Inserting $S(\mathbf{b}) = k = n$ in equation (4.181) we get

$$\lambda^{n-1} \det(\pi^*) = \beta^*_{\mathbf{bb}} = \beta_{\mathbf{bb}} = \lambda^{n-1} \det \pi, \qquad (4.196)$$

$$\lambda^{n-1} \det(\rho^*) = \gamma^*_{\mathbf{bb}} = \gamma_{\mathbf{bb}} = \lambda^{n-1} \det \rho, \qquad (4.197)$$

$$\lambda^{n-1} \det(\hat{\pi}^*) = \hat{\beta}^*_{\mathbf{bb}} = \hat{\beta}_{\mathbf{bb}} = \lambda^{n-1} \det \hat{\pi}, \qquad (4.198)$$

$$\lambda^{n-1} \det(\hat{\rho}^*) = \hat{\gamma}^*_{\mathbf{bb}} = \hat{\gamma}_{\mathbf{bb}} = \lambda^{n-1} \det \hat{\rho}, \qquad (4.199)$$

i. e., equations (4.186) and equations (4.187) hold. ☐

The following Theorem 4.32 and Definition and Theorem 4.33 reveal the close relationship between the dual and the inverse of any algebra isomorphism.

Theorem 4.32. *Let π, ρ, $\hat{\pi}$ and $\hat{\rho}$ represent algebra homomorphisms according to Definition and Theorem 4.14 and 4.14. And let π^*, ρ^*, $\hat{\pi}^*$ and $\hat{\rho}^*$ represent* *their dual algebra homomorphisms according to Definition and Theorem 4.30 respectively. We then have*

$$\pi^*\pi(P_{\mathbf{b}}) = \begin{cases} \beta_{\mathbf{bb}}\,\lambda^{n-1}\det\pi\cdot\mathbf{Z}^+, & 0 = S(\mathbf{b}) = k,\\ \lambda^{n-2}\det\pi\cdot P_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \beta_{\overline{\mathbf{bb}}}\,\lambda^{n-1}\det\pi\cdot\mathbf{I}^+, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.200}$$

$$\pi\pi^*(E'_{\mathbf{b}}) = \begin{cases} \beta_{\mathbf{bb}}\,\lambda^{n-1}\det\pi\cdot(\mathbf{Z}')^-, & 0 = S(\mathbf{b}) = k,\\ \lambda^{n-2}\det\pi\cdot E'_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \beta_{\overline{\mathbf{bb}}}\,\lambda^{n-1}\det\pi\cdot(\mathbf{I}')^-, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.201}$$

$$\rho^*\rho(E_{\mathbf{b}}) = \begin{cases} \gamma_{\mathbf{bb}}\,\mu^{n-1}\det\rho\cdot\mathbf{Z}^-, & 0 = S(\mathbf{b}) = k,\\ \mu^{n-2}\det\rho\cdot E_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \gamma_{\overline{\mathbf{bb}}}\,\mu^{n-1}\det\rho\cdot\mathbf{I}^-, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.202}$$

$$\rho\rho^*(P'_{\mathbf{b}}) = \begin{cases} \gamma_{\mathbf{bb}}\,\mu^{n-1}\det\rho\cdot(\mathbf{Z}')^+, & 0 = S(\mathbf{b}) = k,\\ \mu^{n-2}\det\rho\cdot P'_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \gamma_{\overline{\mathbf{bb}}}\,\mu^{n-1}\det\rho\cdot(\mathbf{I}')^+, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.203}$$

$$\hat\pi^*\hat\pi(P_{\mathbf{b}}) = \begin{cases} \hat\beta_{\mathbf{bb}}\,\lambda^{n-1}\det\hat\pi\cdot\mathbf{Z}^+, & 0 = S(\mathbf{b}) = k,\\ \lambda^{n-2}\det\hat\pi\cdot P_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \hat\beta_{\overline{\mathbf{bb}}}\,\lambda^{n-1}\det\hat\pi\cdot\mathbf{I}^+, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.204}$$

$$\hat\pi\hat\pi^*(P'_{\mathbf{b}}) = \begin{cases} \hat\beta_{\mathbf{bb}}\,\lambda^{n-1}\det\hat\pi\cdot(\mathbf{Z}')^+, & 0 = S(\mathbf{b}) = k,\\ \lambda^{n-2}\det\hat\pi\cdot P'_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \hat\beta_{\overline{\mathbf{bb}}}\,\lambda^{n-1}\det\hat\pi\cdot(\mathbf{I}')^+, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.205}$$

$$\hat\rho^*\hat\rho(E_{\mathbf{b}}) = \begin{cases} \hat\gamma_{\mathbf{bb}}\,\mu^{n-1}\det\hat\rho\cdot\mathbf{Z}^-, & 0 = S(\mathbf{b}) = k,\\ \mu^{n-2}\det\hat\rho\cdot E_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \hat\gamma_{\overline{\mathbf{bb}}}\,\mu^{n-1}\det\hat\rho\cdot\mathbf{I}^-, & n = S(\mathbf{b}) = k, \end{cases} \tag{4.206}$$

$$\hat\rho\hat\rho^*(E'_{\mathbf{b}}) = \begin{cases} \hat\gamma_{\mathbf{bb}}\,\mu^{n-1}\det\hat\rho\cdot(\mathbf{Z}')^-, & 0 = S(\mathbf{b}) = k,\\ \mu^{n-2}\det\hat\rho\cdot E'_{\mathbf{b}}, & 0 < S(\mathbf{b}) = k < n,\\ \hat\gamma_{\overline{\mathbf{bb}}}\,\mu^{n-1}\det\hat\rho\cdot(\mathbf{I}')^-. & n = S(\mathbf{b}) = k, \end{cases} \tag{4.207}$$

Proof. We will prove first equation (4.200) and start with the case $S(\mathbf{b}) = 0$,

$$\begin{aligned} \pi^*\pi(\mathbf{Z}^+) &= \beta_{\mathbf{bb}}\,\pi^*(\mathbf{Z}^+) = \beta_{\mathbf{bb}}\,\pi^*(\mathbf{I}^-) = \beta_{\mathbf{bb}}\,\lambda^{n-1}\det(\pi^*)\cdot\mathbf{I}^- \\ &= \beta_{\mathbf{bb}}\,\lambda^{n-1}\det\pi\cdot\mathbf{Z}^+, \end{aligned} \tag{4.208}$$

continue with the case $S(\mathbf{b}) = n$,

$$\begin{aligned} \pi^*\pi(\mathbf{I}^+) &= \lambda^{n-1}\det\pi\cdot\pi^*(\mathbf{I}^+) = \lambda^{n-1}\det\pi\cdot\pi^*(\mathbf{Z}^-) \\ &= \beta^*_{\overline{\mathbf{bb}}}\,\lambda^{n-1}\det\pi\cdot\mathbf{Z}^- = \beta_{\overline{\mathbf{bb}}}\,\lambda^{n-1}\det\pi\cdot\mathbf{I}^+, \end{aligned} \tag{4.209}$$

and close the prove for equation (4.200) with the remaining case $0 < S(\mathbf{b}) = k < n$,

$$\begin{aligned} \pi^*\pi(P_{\mathbf{b}}) &= \sum_{S(\mathbf{c})=k} \beta_{\mathbf{bc}}\,\pi^*(P'_{\mathbf{c}}) = \sum_{S(\mathbf{c})=k} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\,\beta_{\mathbf{bc}}\,\pi^*(E'_{\overline{\mathbf{c}}}) \\ &= \sum_{\substack{S(\mathbf{c}) = k\\ S(\mathbf{d}) = k}} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\,\beta_{\mathbf{bc}}\,\beta^*_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\,E_{\overline{\mathbf{d}}} = \sum_{\substack{S(\mathbf{c}) = k\\ S(\mathbf{d}) = k}} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\,\alpha_{\overline{\mathbf{d}}\mathbf{d}}\,\beta_{\mathbf{bc}}\,\beta_{\overline{\mathbf{d}}\overline{\mathbf{c}}}\,P_{\mathbf{d}} \end{aligned} \tag{4.210}$$

$$= \lambda^{k-1}\lambda^{n-k-1} \sum_{\substack{S(\mathbf{c})\,=\,k\\ S(\mathbf{d})\,=\,k}} \Bigg[\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{d}}\mathbf{d}} \Bigg(\sum_{\sigma_1} \operatorname{sign}\sigma_1 \prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_1(l)}\mathbf{c}]}\Bigg) \cdot$$

$$\Bigg(\sum_{\sigma_2} \operatorname{sign}\sigma_2 \prod_{m=1}^{n-k} \beta_{[{}_{m}\overline{\mathbf{d}}][{}_{\sigma_2(m)}\overline{\mathbf{c}}]}\Bigg)\Bigg]\; P_{\mathbf{d}}$$

$$= \lambda^{n-2} \sum_{\substack{S(\mathbf{c})\,=\,k\\ S(\mathbf{d})\,=\,k}} \Bigg[\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{d}}\mathbf{d}} \Bigg(\sum_{\sigma_1} \operatorname{sign}\sigma_1 \prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_1(l)}\mathbf{c}]}\Bigg) \cdot$$

$$\Bigg(\sum_{\sigma_3} \operatorname{sign}\sigma_3 \prod_{l=k+1}^{n} \beta_{[{}_{(l-k)}\overline{\mathbf{d}}][{}_{(\sigma_3(l)-k)}\overline{\mathbf{c}}]}\Bigg)\Bigg]\; P_{\mathbf{d}}$$

$$= \lambda^{n-2} \sum_{\substack{S(\mathbf{c})\,=\,k\\ S(\mathbf{d})\,=\,k}} \Bigg[\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{d}}\mathbf{d}} \Bigg(\sum_{\sigma_1} \operatorname{sign}\sigma_1 \prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_{\mathbf{c}}\sigma_1(l)}\mathbf{e}]}\Bigg) \cdot$$

$$\Bigg(\sum_{\sigma_3} \operatorname{sign}\sigma_3 \prod_{l=k+1}^{n} \beta_{[{}_{(l-k)}\overline{\mathbf{d}}][{}_{\sigma_{c}\sigma_3(l)}\mathbf{e}]}\Bigg)\Bigg]\; P_{\mathbf{d}}$$

$$= \lambda^{n-2} \sum_{\substack{S(\mathbf{c})\,=\,k\\ S(\mathbf{d})\,=\,k}} \Bigg[\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{d}}\mathbf{d}} \Bigg(\sum_{\sigma_1} \operatorname{sign}\sigma_1 \prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_{\mathbf{c}}\sigma_4(l)}\mathbf{e}]}\Bigg) \cdot$$

$$\Bigg(\sum_{\sigma_3} \operatorname{sign}\sigma_3 \prod_{l=k+1}^{n} \beta_{[{}_{(l-k)}\overline{\mathbf{d}}][{}_{\sigma_{c}\sigma_4(l)}\mathbf{e}]}\Bigg)\Bigg]\; P_{\mathbf{d}}$$

$$= \lambda^{n-2} \sum_{\substack{S(\mathbf{c})\,=\,k\\ S(\mathbf{d})\,=\,k}} \Bigg[\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\mathbf{c}\overline{\mathbf{c}}}\, \alpha_{\overline{\mathbf{d}}\mathbf{d}} \Bigg(\sum_{\sigma_1,\sigma_3} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\; \operatorname{sign}\sigma_1\; \operatorname{sign}\sigma_3 \cdot$$

$$\prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_{\mathbf{c}}\sigma_4(l)}\mathbf{e}]} \prod_{l=k+1}^{n} \beta_{[{}_{(l-k)}\overline{\mathbf{d}}][{}_{\sigma_{c}\sigma_4(l)}\mathbf{e}]}\Bigg)\Bigg]\; P_{\mathbf{d}}$$

$$= \lambda^{n-2}\,(-1)^{k(n-k)} \sum_{S(\mathbf{d})=k} \Bigg[\alpha_{\overline{\mathbf{d}}\mathbf{d}} \Bigg(\sum_{\sigma_5} \operatorname{sign}\sigma_5 \prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_5(l)}\mathbf{e}]} \cdot$$

$$\prod_{l=k+1}^{n} \beta_{[{}_{(l-k)}\overline{\mathbf{d}}][{}_{\sigma_5(l)}\mathbf{e}]}\Bigg)\Bigg]\; P_{\mathbf{d}}$$

$$= \lambda^{n-2}\,(-1)^{k(n-k)} \Bigg[\alpha_{\overline{\mathbf{b}}\mathbf{b}} \Bigg(\sum_{\sigma_5} \operatorname{sign}\sigma_5 \prod_{l=1}^{k} \beta_{[{}_{l}\mathbf{b}][{}_{\sigma_5(l)}\mathbf{e}]} \cdot$$

$$\prod_{l=k+1}^{n} \beta_{[{}_{(l-k)}\overline{\mathbf{b}}][{}_{\sigma_5(l)}\mathbf{e}]}\Bigg)\Bigg]\; P_{\mathbf{b}}$$

$$
\begin{aligned}
&= \lambda^{n-2}\,(-1)^{k(n-k)} \left[\alpha_{\overline{\mathbf{b}}\mathbf{b}} \left(\sum_{\sigma_5} \operatorname{sign}\sigma_5 \prod_{l=1}^{n} \beta_{[{}_{\sigma_{\mathbf{b}}(l)}\mathbf{e}][{}_{\sigma_5(l)}\mathbf{e}]} \right)\right] P_{\mathbf{b}} \\
&= \lambda^{n-2}\,(-1)^{k(n-k)} \left[\alpha_{\overline{\mathbf{b}}\mathbf{b}}\alpha_{\mathbf{b}\overline{\mathbf{b}}} \left(\sum_{\sigma_5} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \operatorname{sign}\sigma_5 \cdot \right.\right. \\
&\qquad \left.\left. \prod_{m=1}^{n} \beta_{[{}_{m}\mathbf{e}][{}_{\sigma_5\sigma_{\mathbf{b}}^{-1}(m)}\mathbf{e}]} \right)\right] P_{\mathbf{b}} \\
&= \lambda^{n-2}\,(-1)^{2k(n-k)} \left[\sum_{\sigma} \operatorname{sign}\sigma \prod_{m=1}^{n} \beta_{[{}_{m}\mathbf{e}][{}_{\sigma(m)}\mathbf{e}]} \right] P_{\mathbf{b}} \\
&= \lambda^{n-2} \det\pi \cdot P_{\mathbf{b}},
\end{aligned}
$$

where we used

$$
\begin{aligned}
\sigma_1 :\ & \{1,2,\dots,k\} && \longrightarrow && \{1,2,\dots,k\} \\
& l && \longmapsto && \sigma_1(l),
\end{aligned} \tag{4.211}
$$

$$
\begin{aligned}
\sigma_2 :\ & \{1,2,\dots,n-k\} && \longrightarrow && \{1,2,\dots,n-k\} \\
& m && \longmapsto && \sigma_2(m),
\end{aligned} \tag{4.212}
$$

$$
\begin{aligned}
\sigma_3 :\ & \{k+1,k+2,\dots,n\} && \longrightarrow && \{k+1,k+2,\dots,n\} \\
& l && \longmapsto && \sigma_3(l) := \sigma_2(l-k)+k,
\end{aligned} \tag{4.213}
$$

$$
S(\mathbf{e}) = n, \quad S(\mathbf{f}) = k, \tag{4.214}
$$

$$
\begin{aligned}
\sigma_{\mathbf{f}} :\ & \{1,2,\dots,n\} && \longrightarrow && \{1,2,\dots,n\} \\
& m && \longmapsto && \sigma_{\mathbf{f}}(m)
\end{aligned} \tag{4.215}
$$

with

$$
{}_{\sigma_{\mathbf{f}}(m)}\mathbf{e} = {}_{m}\mathbf{f} \qquad \text{for} \qquad 1 \le m \le k, \tag{4.216}
$$

$$
{}_{\sigma_{\mathbf{f}}(m)}\mathbf{e} = {}_{(m-k)}\overline{\mathbf{f}} \qquad \text{for} \qquad k+1 \le m \le n, \tag{4.217}
$$

$$
\begin{aligned}
\sigma_4 :\ & \{1,2,\dots,n\} && \longrightarrow && \{1,2,\dots,n\} \\
& l && \longmapsto && \sigma_4(l) := \sigma_1\sigma_3(l)
\end{aligned} \tag{4.218}
$$

with

$$
\sigma_3(l) := l \qquad \text{for} \qquad 1 \le l \le k, \tag{4.219}
$$

$$
\sigma_1(l) := l \qquad \text{for} \qquad k+1 \le l \le n, \tag{4.220}
$$

$$
\begin{aligned}
\sigma_5 :\ & \{1,2,\dots,n\} && \longrightarrow && \{1,2,\dots,n\} \\
& l && \longmapsto && \sigma_5(l) := \sigma_{\mathbf{c}}\sigma_4(l)
\end{aligned} \tag{4.221}
$$

$$
\begin{aligned}
\sigma :\ & \{1,2,\dots,n\} && \longrightarrow && \{1,2,\dots,n\} \\
& l && \longmapsto && \sigma(l) := \sigma_5\sigma_{\mathbf{b}}^{-1}(l)
\end{aligned} \tag{4.222}
$$

and

$$
\operatorname{sign}\sigma_3 = \prod_{k+1 \le i < j \le n} \frac{\sigma_3(j)-\sigma_3(i)}{j-i} \tag{4.223}
$$

$$
\begin{aligned}
&= \prod_{k+1\le i<j\le n} \frac{\sigma_2(j-k)-\sigma_2(i-k)}{j-i} = \prod_{1\le i'<j'\le n} \frac{\sigma_2(j')-\sigma_2(i')}{j'-i'} \\
&= \operatorname{sign}\sigma_2, \qquad i' := i-k, \quad j' := j-k,
\end{aligned}
$$

$$\operatorname{sign}\sigma_{\mathbf{b}} = \alpha_{\mathbf{b}\overline{\mathbf{b}}}, \quad \operatorname{sign}\sigma_{\mathbf{c}} = \alpha_{\mathbf{c}\overline{\mathbf{c}}}, \tag{4.224}$$

$$\operatorname{sign}\sigma_4 = \operatorname{sign}\sigma_1 \cdot \operatorname{sign}\sigma_3, \tag{4.225}$$

$$\operatorname{sign}\sigma_5 = \alpha_{\mathbf{c}\overline{\mathbf{c}}} \cdot \operatorname{sign}\sigma_1 \cdot \operatorname{sign}\sigma_3, \tag{4.226}$$

$$\operatorname{sign}\sigma = \alpha_{\mathbf{b}\overline{\mathbf{b}}} \cdot \operatorname{sign}\sigma_5. \tag{4.227}$$

The case $0 < S(\mathbf{b}) = k < n$ for equation (4.201) is proven along similar steps as in equation (4.210),

$$
\begin{aligned}
\pi\pi^*(E'_{\mathbf{b}}) &= \sum_{S(\mathbf{c})=k} \beta^*_{\mathbf{bc}}\, \pi(E_{\mathbf{c}}) = \sum_{S(\mathbf{c})=k} \alpha_{\mathbf{c}\overline{\mathbf{c}}}\, \beta^*_{\mathbf{bc}}\, \pi(P_{\overline{\mathbf{c}}}) \qquad (4.228)\\
&= \sum_{\substack{S(\mathbf{c})=k\\ S(\mathbf{d})=k}} \alpha_{\mathbf{c}\overline{\mathbf{c}}}\, \beta^*_{\mathbf{bc}}\, \beta_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\, P'_{\overline{\mathbf{d}}} = \sum_{\substack{S(\mathbf{c})=k\\ S(\mathbf{d})=k}} \alpha_{\mathbf{c}\overline{\mathbf{c}}}\, \alpha_{\mathbf{d}\overline{\mathbf{d}}}\, \beta_{\mathbf{cb}}\, \beta_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\, E'_{\mathbf{d}} \\
&= \lambda^{n-2} \det\pi \cdot E'_{\mathbf{b}}.
\end{aligned}
$$

The equations (4.202) to (4.207) are proven as well along similar steps. □

Definition and Theorem 4.33 (0-Vectors). According to Definition and Theorem 4.14, 4.15 and Theorem 4.32, we may choose the transformations of 0-vectors by the different types of algebra homomorphisms to be[11]

$$\pi(\mathbf{Z}^+) := \frac{1-\delta_{0\,\det\pi}}{\lambda}(\mathbf{Z}')^+, \qquad \rho(\mathbf{Z}^-) := \frac{1-\delta_{0\,\det\rho}}{\mu}(\mathbf{Z}')^-, \tag{4.229}$$

$$\hat{\pi}(\mathbf{Z}^+) := \frac{1-\delta_{0\,\det\hat{\pi}}}{\lambda}(\mathbf{Z}')^-, \qquad \hat{\rho}(\mathbf{Z}^-) := \frac{1-\delta_{0\,\det\hat{\rho}}}{\mu}(\mathbf{Z}')^+, \tag{4.230}$$

i. e., the coefficients from equations (4.48), (4.52), (4.74) and (4.78) take on the values

$$\beta_{\mathbf{bb}} = \frac{1-\delta_{0\,\det\pi}}{\lambda}, \qquad \gamma_{\mathbf{bb}} = \frac{1-\delta_{0\,\det\rho}}{\mu}, \tag{4.231}$$

$$\hat{\beta}_{\mathbf{bb}} = \frac{1-\delta_{0\,\det\hat{\pi}}}{\lambda}, \qquad \hat{\gamma}_{\mathbf{bb}} = \frac{1-\delta_{0\,\det\hat{\rho}}}{\mu}, \tag{4.232}$$

where the binary index $\mathbf{b}$ satisfies $S(\mathbf{b}) = 0$. With this choice, the formulas of equations (4.200) to (4.207) in Theorem 4.32 simplify to

$$\pi^*\pi(P_{\mathbf{b}}) = \lambda^{n-2}\det\pi \cdot P_{\mathbf{b}}, \qquad 0 \le S(\mathbf{b}) = k \le n,, \tag{4.233}$$

---

[11] Why do we use the Kronecker-delta-function in the defining equations (4.229) and (4.230)? The main reason is to include the vanishing algebra homomorphisms, i. e. for example $\pi(X) = 0$ for all $X \in \Lambda_n$. With a definition like $\pi(\mathbf{Z}^+) = {}^{(\mathbf{Z}')^+}\!/\!_{\lambda}$ there would be no vanishing π possible. With the chosen definitions in equations (4.229) and (4.230) the algebra homomorphisms behave similarily with respect to 0-vectors as they do with respect to n-vectors. They vanish, if the algebra homomorphism has a vanishing determinant, otherwise not. Another reason for this choice of definition is Theorem 4.34. The assertion of the latter would be broken for the 0-vectors without using the Kronecker-delta-function in the defining equations (4.229) and (4.230).

$$\pi\pi^*(E'_{\mathbf{b}}) = \lambda^{n-2}\det\pi\cdot E'_{\mathbf{b}}, \tag{4.234}$$

$$\rho^*\rho(E_{\mathbf{b}}) = \mu^{n-2}\det\rho\cdot E_{\mathbf{b}}, \tag{4.235}$$

$$\rho\rho^*(P'_{\mathbf{b}}) = \mu^{n-2}\det\rho\cdot P'_{\mathbf{b}}, \tag{4.236}$$

$$\hat{\pi}^*\hat{\pi}(P_{\mathbf{b}}) = \lambda^{n-2}\det\hat{\pi}\cdot P_{\mathbf{b}}, \tag{4.237}$$

$$\hat{\pi}\hat{\pi}^*(P'_{\mathbf{b}}) = \lambda^{n-2}\det\hat{\pi}\cdot P'_{\mathbf{b}}, \tag{4.238}$$

$$\hat{\rho}^*\hat{\rho}(E_{\mathbf{b}}) = \mu^{n-2}\det\hat{\rho}\cdot E_{\mathbf{b}}, \tag{4.239}$$

$$\hat{\rho}\hat{\rho}^*(E'_{\mathbf{b}}) = \mu^{n-2}\det\hat{\rho}\cdot E'_{\mathbf{b}}. \tag{4.240}$$

Proof. We are free to define the algebra homomorphism as done in equations (4.229) and (4.230). There will be no conflict with no algebra homomorphism. As a consequence we get the coefficients of equations (4.231) and (4.232).

In oder to get equations (4.233) to (4.240), insert the the coefficients of equations (4.231) and (4.232) in equations (4.200) to (4.207) respectively. □

Theorem 4.34. *For any degenerate algebra homomorpohism ϕ according to Notation 4.22 there is at least one $X \in \Lambda_n^k \setminus \{\mathbf{0}\}$ for all $k \in \{0, 1, 2, \ldots, n\}$ with $\phi(X) = \mathbf{0}$.*

Proof. In case $k = n$, we get for $X = \xi\mathbf{I} \in \Lambda_n^n \setminus \{\mathbf{0}\}$ $\phi(X) = \xi\phi(\mathbf{I}) = \xi\nu^{n-1}\det\phi\cdot\mathbf{I} = \mathbf{0}$, since $\det\phi = 0$.

In case $k = 0$, we get for $X = \xi\mathbf{Z} \in \Lambda_n^0\setminus\{\mathbf{0}\}$ $\phi(X) = \xi\phi(\mathbf{Z}) = \mathbf{0}$, because of Definition and Theorem 4.33.

The degenerate algebra homomorphism ϕ restricted to Λ_n^k with $k \in \{1, 2, \ldots, n-1\}$ is not injective, i.e., there are elements $X_1 \in \Lambda_n^k \setminus \{\mathbf{0}\}$ and $X_2 \in \Lambda_n^k\setminus\{\mathbf{0}\}$ with $X_1 \neq X_2$ and $\phi(X_1) = \phi(X_2)$. Thus, with $X := X_1 - X_2 \neq \mathbf{0}$ we get $\phi(X) = \phi(X_1 - X_2) = \phi(X_1) - \phi(X_2) = \mathbf{0}$. □

Let ϕ be an algebra homomorphism, ϕ^* its dual transformation and $\mathbf{b}$ a binary index with check sum $S(\mathbf{b}) = k$, then the product $\phi^*\phi(B_{\mathbf{b}})$ is represented by the product of the cofactor matrix $\operatorname{cof}\big(\underline{K_{\overline{k}}}\big)$ and the matrix $\underline{K_{\overline{k}}}$. The matrix $\underline{K_{\overline{k}}}$ represents the transformation of k-vectors by ϕ. Corollary 4.36 will provide the details. Since the concept of a complementary basis in an exterior algebra will be used in Corollary 4.36, the corresponding notation is introduced now.

Notation 4.35 (Complementary Basis of an Exterior Algebra). In order to make the transition from a given basis $\{P_{\mathbf{b}}\}$, $\{E_{\mathbf{b}}\}$ or $\{B_{\mathbf{b}}\}$ to its complementary basis, we use the following notation:

$$P^*_{\mathbf{b}} := E_{\mathbf{b}}, \qquad E^*_{\mathbf{b}} := P_{\mathbf{b}}, \tag{4.241}$$

$$B^*_{\mathbf{b}} := \begin{cases} P_{\mathbf{b}}, & \text{in case of } B_{\mathbf{b}} = E_{\mathbf{b}}, \\ E_{\mathbf{b}}, & \text{in case of } B_{\mathbf{b}} = P_{\mathbf{b}}. \end{cases} \tag{4.242}$$

Corollary 4.36 (Cofactor Matrix, Adjugate, Classical Adjoint). *Let ϕ denote a generic algebra homomorphism according to Notation 4.22 and let*

$$\phi^* : \quad \Lambda_n^* \qquad \longleftarrow \qquad (\Lambda'_n)^* \tag{4.243}$$

$$\phi^*((B'_{\mathbf{b}})^*) = \sum_{S(\mathbf{c})=k} \kappa^*_{\mathbf{bc}} B^*_{\mathbf{c}} \qquad \longleftarrow\!\!\!\mid \qquad (B'_{\mathbf{b}})^*$$

be its dual algebra homomorphism with $S(\mathbf{b}) = k$*. According to Definition and Theorem 4.33 and the proof of Theorem 4.32 – compare equations (4.210) and (4.228) – we have*

$$\phi^*\phi(B_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \kappa_{\mathbf{bc}} \left(\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{b}}\mathbf{b}}\, \kappa_{\overline{\mathbf{c}}\overline{\mathbf{b}}}\right)^T B_{\mathbf{b}} = \nu^{n-2} \det\phi \cdot B_{\mathbf{b}}, \tag{4.244}$$

where

$$\operatorname{cof}\left(\underline{K_{\overline{k}}}\right) := \left(\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{b}}\mathbf{b}}\, \kappa_{\overline{\mathbf{c}}\overline{\mathbf{b}}}\right)_{S(\mathbf{c})=k; S(\mathbf{b})=k} \tag{4.245}$$

is the cofactor matrix of order k *of the matrix* $\underline{K_{\overline{k}}}$ *and*

$$\operatorname{adj}\left(\underline{K_{\overline{k}}}\right) := \operatorname{cof}\left(\underline{K_{\overline{k}}}\right)^T \tag{4.246}$$

is the respective adjugate matrix of order k*. In case of order* $k = 1$*, the expressions of equations (4.245) and (4.246) reduce to the* classical cofactor matrix $\operatorname{cof}(\underline{K_{\overline{1}}})$ *and the* classical adjoint $\operatorname{adj}(\underline{K_{\overline{1}}})$*, i. e. with the classical indices* $i, j \in \{1, \ldots, n\} \subset \mathbb{N}$ *related to the binary indices* $\mathbf{c}$ *and* $\mathbf{b}$ *by*

$${}_i\mathbf{d} := \mathbf{c}, \qquad {}_j\mathbf{d} := \mathbf{b}, \qquad\qquad S(\mathbf{d}) = n, \tag{4.247}$$

we get

$$\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{b}}\mathbf{b}} = (-1)^{i+j}, \tag{4.248}$$

$$\kappa_{\overline{\mathbf{c}}\overline{\mathbf{b}}} = \nu^{n-2} \sum_{\sigma} \operatorname{sign}\sigma \prod_{l=1}^{n-1} \kappa_{[{}_l\overline{\mathbf{c}}][{}_{\sigma(l)}\overline{\mathbf{b}}]} \tag{4.249}$$

$$= \nu^{n-2} \det \begin{pmatrix} \kappa_{[{}_1\mathbf{d}][{}_1\mathbf{d}]} & \cdots & \kappa_{[{}_1\mathbf{d}][{}_{j-1}\mathbf{d}]} & \kappa_{[{}_1\mathbf{d}][{}_{j+1}\mathbf{d}]} & \cdots & \kappa_{[{}_1\mathbf{d}][{}_n\mathbf{d}]} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{[{}_{i-1}\mathbf{d}][{}_1\mathbf{d}]} & \cdots & \kappa_{[{}_{i-1}\mathbf{d}][{}_{j-1}\mathbf{d}]} & \kappa_{[{}_{i-1}\mathbf{d}][{}_{j+1}\mathbf{d}]} & \cdots & \kappa_{[{}_{i-1}\mathbf{d}][{}_n\mathbf{d}]} \\ \kappa_{[{}_{i+1}\mathbf{d}][{}_1\mathbf{d}]} & \cdots & \kappa_{[{}_{i+1}\mathbf{d}][{}_{j-1}\mathbf{d}]} & \kappa_{[{}_{i+1}\mathbf{d}][{}_{j+1}\mathbf{d}]} & \cdots & \kappa_{[{}_{i+1}\mathbf{d}][{}_n\mathbf{d}]} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{[{}_n\mathbf{d}][{}_1\mathbf{d}]} & \cdots & \kappa_{[{}_n\mathbf{d}][{}_{j-1}\mathbf{d}]} & \kappa_{[{}_n\mathbf{d}][{}_{j+1}\mathbf{d}]} & \cdots & \kappa_{[{}_n\mathbf{d}][{}_n\mathbf{d}]} \end{pmatrix}.$$

Proof. Using the expressions of equation (4.148) and (4.147), we can show equation (4.248),

$$\alpha_{\overline{\mathbf{c}}\mathbf{c}}\, \alpha_{\overline{\mathbf{b}}\mathbf{b}} = \alpha_{\mathbf{c}\overline{\mathbf{c}}}\, \alpha_{\mathbf{b}\overline{\mathbf{b}}} = (-1)^{(i-1)+(j-1)} = (-1)^{i+j-2} = (-1)^{i+j}. \tag{4.250}$$

□

Corollary 4.37 (Two Representations for the Inverse Algebra Isomorphisms). *Let* ϕ *denote a generic algebra isomorphism (*$\det\phi \neq 0$*) according to Notation 4.22. Then, with Theorem 4.23,* ϕ^{-1} *represents an inverse of* ϕ*. With Definition and Theorem 4.33, we now have another representation of the same inverse algebra isomorphism,*

$$\phi^{-1} := \frac{\phi^*}{\nu^{n-2}\det\phi}. \tag{4.251}$$

The two representations of the inverse of an algebra isomorphism ϕ are listed below for each type of algebra isomorphism. For an algebra isomorphism π in the plus approach,

$$\pi(X \wedge Y) = \lambda \cdot \pi(X) \wedge \pi(Y), \tag{4.252}$$

the two representations of the inverse algebra isomorphism π^{-1} prteserve the exterior products as follows

$$\pi^{-1}(X' \wedge Y') = \lambda^{-1} \cdot \pi^{-1}(X') \wedge \pi^{-1}(Y'), \tag{4.253}$$

$$\pi^{-1}(X' \vee Y') = \lambda^{n-1} \det \pi \cdot \pi^{-1}(X') \vee \pi^{-1}(Y'); \tag{4.254}$$

for an algebra isomorphism ρ in the minus approach,

$$\rho(X \vee Y) = \mu \cdot \rho(X) \vee \rho(Y), \tag{4.255}$$

the two representations of the inverse algebra isomorphism ρ^{-1} preserve the exterior products as follows

$$\rho^{-1}(X' \vee Y') = \mu^{-1} \cdot \rho^{-1}(X') \vee \rho^{-1}(Y'), \tag{4.256}$$

$$\rho^{-1}(X' \wedge Y') = \mu^{n-1} \det \rho \cdot \rho^{-1}(X') \wedge \rho^{-1}(Y'); \tag{4.257}$$

for an algebra isomorphism $\hat{\pi}$ from the plus to the minus approach,

$$\hat{\pi}(X \wedge Y) = \lambda \cdot \hat{\pi}(X) \vee \hat{\pi}(Y), \tag{4.258}$$

the two representations of the inverse algebra isomorphism $\hat{\pi}^{-1}$ preserve the exterior products as follows

$$\hat{\pi}^{-1}(X' \vee Y') = \lambda^{-1} \cdot \hat{\pi}^{-1}(X') \wedge \hat{\pi}^{-1}(Y'), \tag{4.259}$$

$$\hat{\pi}^{-1}(X' \wedge Y') = \lambda^{n-1} \det \hat{\pi} \cdot \hat{\pi}^{-1}(X') \vee \hat{\pi}^{-1}(Y'); \tag{4.260}$$

and for an algebra isomorphism $\hat{\rho}$ from the minus to the plus approach,

$$\hat{\rho}(X \vee Y) = \mu \cdot \hat{\rho}(X) \wedge \hat{\rho}(Y), \tag{4.261}$$

the two representations of the inverse algebra isomorphism $\hat{\rho}^{-1}$ preserve the exterior products as follows

$$\hat{\rho}^{-1}(X' \wedge Y') = \mu^{-1} \cdot \hat{\rho}^{-1}(X') \vee \hat{\rho}^{-1}(Y'), \tag{4.262}$$

$$\hat{\rho}^{-1}(X' \vee Y') = \mu^{n-1} \det \hat{\rho} \cdot \hat{\rho}^{-1}(X') \wedge \hat{\rho}^{-1}(Y'). \tag{4.263}$$

Proof. Equation (4.251) is a consequence of Definition and Theorem 4.33 in case ϕ represents an isomorphism.

Equation (4.252) is by definition true for an algebra isomorphism π. Compare equation (4.50). The first representation of the inverse algebra isomorphism π^{-1} preserves the exterior products according to equation (4.253). This is a consequence of Theorem (4.23) applied to equation (4.252). The second representation of the inverse algebra isomorphism π^{-1} preserves the exterior products according to equation (4.254). This is a consequence of inserting equation (4.251) into equation (4.182) from Definition and Theorem (4.31).

The equations (4.255) to (4.263) for the other types of algebra isomorphisms follow with similar arguments. □

It is now not difficult to conclude, any algebra isomorphism is preserving *both* exterior products. Since the preservation of the exterior products is essential for projectivities later on, we highlight the corresponding property of generic algebra isomorphisms in

Theorem 4.38. *Let ϕ denote a generic algebra isomorphism (*$\det\phi \neq 0$*) according to Notation 4.22. Then ϕ preserves both exterior products,*

$$\phi \text{ even}: \qquad \pi(X \wedge Y) = \lambda \cdot \pi(X) \wedge \pi(Y), \tag{4.264}$$

$$\pi(X \vee Y) = \frac{1}{\lambda^{n-1}\det\pi} \cdot \pi(X) \vee \pi(Y), \tag{4.265}$$

$$\rho(X \vee Y) = \mu \cdot \rho(X) \vee \rho(Y), \tag{4.266}$$

$$\rho(X \wedge Y) = \frac{1}{\mu^{n-1}\det\rho} \cdot \rho(X) \wedge \rho(Y), \tag{4.267}$$

$$\phi \text{ odd}: \qquad \hat{\pi}(X \wedge Y) = \lambda \cdot \hat{\pi}(X) \vee \hat{\pi}(Y), \tag{4.268}$$

$$\hat{\pi}(X \vee Y) = \frac{1}{\lambda^{n-1}\det\hat{\pi}} \cdot \hat{\pi}(X) \wedge \hat{\pi}(Y), \tag{4.269}$$

$$\hat{\rho}(X \vee Y) = \mu \cdot \hat{\rho}(X) \wedge \hat{\rho}(Y), \tag{4.270}$$

$$\hat{\rho}(X \wedge Y) = \frac{1}{\mu^{n-1}\det\hat{\rho}} \cdot \hat{\rho}(X) \vee \hat{\rho}(Y). \tag{4.271}$$

Proof. Equations (4.264), (4.266), (4.268) and (4.270) are true by definition. Compare (4.50), (4.54), (4.76) and (4.80) of Definition and Theorem 4.14 and 4.15. We get equations (4.265), (4.267), (4.269) and (4.271) by applying Theorem 4.23 to the equations (4.254), (4.257), (4.260) and (4.263) of Corollary 4.37. □

4.5. One and the Same Algebra Isomorphism in Different Approaches

The two representations mentioned in Corollary 4.37 and displayed in Theorem 4.38 are either an isomorphism in the plus approach and an isomorphism in the minus approach (in case of even isomorphisms) or an isomorphism from the plus to the minus approach and an isomorphism from the minus to the plus approach (in case of odd isomorphisms). In general, any pair of two different even or odd algebra isomorphisms respectively, which represent the same transformation, are related as follows:

Theorem 4.39. *The algebra isomorphism π in the plus approach,*

$$\pi(P_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \beta_{\mathbf{bc}} P'_{\mathbf{c}}, \qquad 0 \leq S(\mathbf{b}) = k \leq n, \tag{4.272}$$

and the algebra isomorphism ρ in the minus approach,

$$\rho(E_{\mathbf{d}}) = \sum_{S(\mathbf{e})=m} \gamma_{\mathbf{de}} E'_{\mathbf{e}}, \qquad 0 \leq S(\mathbf{d}) = m \leq n, \tag{4.273}$$

defined in Definition an Theorem 4.14, represent one and the same one-to-one transformation, if and only if

$$\gamma_{\mathbf{de}} = \alpha_{\mathbf{d}\overline{\mathbf{d}}}\,\beta_{\overline{\mathbf{d}}\overline{\mathbf{e}}}\,\alpha_{\mathbf{e}\overline{\mathbf{e}}} \tag{4.274}$$

or, equivalently, if and only if

$$\beta_{\mathbf{bc}} = \alpha_{\overline{\mathbf{b}}\mathbf{b}}\,\gamma_{\overline{\mathbf{b}}\overline{\mathbf{c}}}\,\alpha_{\overline{\mathbf{c}}\mathbf{c}}. \tag{4.275}$$

In addition we have

$$\det\pi = \frac{1}{\mu\,\lambda^{n-1}}, \qquad \det\rho = \frac{1}{\lambda\,\mu^{n-1}}, \qquad \frac{\det\pi}{\det\rho} = \left(\frac{\mu}{\lambda}\right)^{n-2}. \tag{4.276}$$

The algebra isomorphism $\hat{\pi}$ from the plus to the minus approach,

$$\hat{\pi}(P_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \hat{\beta}_{\mathbf{bc}} E'_{\mathbf{c}}, \qquad 0 \le S(\mathbf{b}) = k \le n, \tag{4.277}$$

and the algebra isomorphism $\hat{\rho}$ from the minus to the plus approach,

$$\hat{\rho}(E_{\mathbf{d}}) = \sum_{S(\mathbf{e})=m} \hat{\gamma}_{\mathbf{de}} P'_{\mathbf{e}}, \qquad 0 \le S(\mathbf{d}) = m \le n, \tag{4.278}$$

defined in Definition an Theorem 4.15, represent one and the same one-to-one transformation, if and only if

$$\hat{\gamma}_{\mathbf{de}} = \alpha_{\mathbf{d}\overline{\mathbf{d}}}\,\hat{\beta}_{\overline{\mathbf{d}}\overline{\mathbf{e}}}\,\alpha_{\overline{\mathbf{e}}\mathbf{e}} \tag{4.279}$$

or, equivalently, if and only if

$$\hat{\beta}_{\mathbf{bc}} = \alpha_{\overline{\mathbf{b}}\mathbf{b}}\,\hat{\gamma}_{\overline{\mathbf{b}}\overline{\mathbf{c}}}\,\alpha_{\mathbf{c}\overline{\mathbf{c}}}. \tag{4.280}$$

In addition we have

$$\det\hat{\pi} = \frac{1}{\mu\,\lambda^{n-1}}, \qquad \det\hat{\rho} = \frac{1}{\lambda\,\mu^{n-1}}, \qquad \frac{\det\hat{\pi}}{\det\hat{\rho}} = \left(\frac{\mu}{\lambda}\right)^{n-2}. \tag{4.281}$$

Proof. Using the transformations from the basis in the minus to the basis in the plus approach of equation (4.146), $P_{\mathbf{b}} = \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}}$ and $P'_{\mathbf{c}} = \alpha_{\overline{\mathbf{c}}\mathbf{c}} E'_{\overline{\mathbf{c}}}$, we get from equation (4.272) the following expression,

$$\pi(E_{\overline{\mathbf{b}}}) = \sum_{S(\mathbf{c})=k} \alpha_{\overline{\mathbf{b}}\mathbf{b}}\beta_{\mathbf{bc}}\alpha_{\overline{\mathbf{c}}\mathbf{c}} E'_{\overline{\mathbf{c}}}, \tag{4.282}$$

and with $\overline{\mathbf{b}} = \mathbf{d}$, $\overline{\mathbf{c}} = \mathbf{e}$ and $m = n - k$,

$$\pi(E_{\mathbf{d}}) = \sum_{S(\mathbf{e})=n-k} \alpha_{\mathbf{d}\overline{\mathbf{d}}}\beta_{\overline{\mathbf{d}}\overline{\mathbf{e}}}\alpha_{\mathbf{e}\overline{\mathbf{e}}} E'_{\mathbf{e}}, \tag{4.283}$$

and thus

$$\rho(E_{\mathbf{d}}) = \sum_{S(\mathbf{e})=m} \gamma_{\mathbf{de}} E'_{\mathbf{e}}, \qquad \text{with} \qquad \gamma_{\mathbf{de}} = \alpha_{\mathbf{d}\overline{\mathbf{d}}}\,\beta_{\overline{\mathbf{d}}\overline{\mathbf{e}}}\,\alpha_{\mathbf{e}\overline{\mathbf{e}}}. \tag{4.284}$$

The equations (4.274) and (4.275) are equivalent — not only numerically, but also in the sense, the coefficients $\gamma_{\mathbf{de}}$ represent an algebra isomorphism if and only if the coefficients $\beta_{\mathbf{bc}}$ represent an algebra isomorphism. In order to show this, we first assume π to be an algebra isomorphism. With equation (4.254) we get for $A, B \in \Lambda_n^-$

$$\begin{aligned}\rho(A \vee B) &= \rho\big(\pi^{-1}\rho(A) \vee \pi^{-1}\rho(B)\big) \\ &= \frac{1}{\lambda^{n-1}\det\pi}\rho\pi^{-1}(\rho(A) \vee \rho(B)) \\ &= \mu \cdot \rho(A) \vee \rho(B) \quad \text{with} \quad \mu = \frac{1}{\lambda^{n-1}\det\pi},\end{aligned} \tag{4.285}$$

i. e., the transformation ρ preserves the minor exterior product $\vee$ and represents an algebra isomorphism in the minus approach. With equation (4.285) and

$$\begin{aligned}\mathbf{I}^- &= \pi^{-1}\rho(\mathbf{I}^-) = \mu^{n-1}\det\rho \cdot \pi^{-1}(\mathbf{I}^-) = \mu^{n-1}\det\rho \cdot \pi^{-1}(\mathbf{Z}^+) \\ &= \lambda\mu^{n-1}\det\rho \cdot \mathbf{Z}^+ = \lambda\mu^{n-1}\det\rho \cdot \mathbf{I}^-\end{aligned} \tag{4.286}$$

we get equations (4.276).

Secondly we assume ρ to be an algebra isomorphism. With equation (4.257) we get for $A, B \in \Lambda_n^+$

$$\begin{aligned}\pi(A \wedge B) &= \pi\big(\rho^{-1}\pi(A) \wedge \rho^{-1}\pi(B)\big) \\ &= \frac{1}{\mu^{n-1}\det\rho}\pi\rho^{-1}(\pi(A) \wedge \pi(B)) \\ &= \lambda \cdot \pi(A) \wedge \pi(B) \quad \text{with} \quad \lambda = \frac{1}{\mu^{n-1}\det\rho},\end{aligned} \tag{4.287}$$

i. e., the transformation π preserves the major exterior product $\wedge$ and represents an algebra isomorphism in the plus approach. With equation (4.287) and

$$\begin{aligned}\mathbf{I}^+ &= \rho^{-1}\pi(\mathbf{I}^+) = \lambda^{n-1}\det\pi \cdot \rho^{-1}(\mathbf{I}^+) = \lambda^{n-1}\det\pi \cdot \rho^{-1}(\mathbf{Z}^-) \\ &= \mu\lambda^{n-1}\det\pi \cdot \mathbf{Z}^- = \mu\lambda^{n-1}\det\pi \cdot \mathbf{I}^+\end{aligned} \tag{4.288}$$

we get equations (4.276) again.

Using the transformations from the basis in the minus to the basis in the plus approach of equation (4.146), $P_{\mathbf{b}} = \alpha_{\overline{\mathbf{b}}\mathbf{b}}E_{\overline{\mathbf{b}}}$, and using the transformations from the basis in the plus to the basis in the minus approach of equation (4.145) $E'_{\mathbf{c}} = \alpha_{\mathbf{c}\overline{\mathbf{c}}}P'_{\overline{\mathbf{c}}}$, we get from equation (4.276) the following expression,

$$\hat{\pi}(E_{\overline{\mathbf{b}}}) = \sum_{S(\mathbf{c})=k} \alpha_{\overline{\mathbf{b}}\mathbf{b}}\hat{\beta}_{\mathbf{b}\mathbf{c}}\alpha_{\mathbf{c}\overline{\mathbf{c}}}P'_{\overline{\mathbf{c}}}, \tag{4.289}$$

and with $\overline{\mathbf{d}} = \mathbf{b}$, $\overline{\mathbf{e}} = \mathbf{c}$ and $m = n - k$,

$$\hat{\pi}(E_{\mathbf{d}}) = \sum_{S(\mathbf{e})=n-k} \alpha_{\mathbf{d}\overline{\mathbf{d}}}\hat{\beta}_{\overline{\mathbf{d}}\overline{\mathbf{e}}}\alpha_{\overline{\mathbf{e}}\mathbf{e}}P'_{\mathbf{e}}, \tag{4.290}$$

and thus

$$\hat{\rho}(E_{\mathbf{d}}) = \sum_{S(\mathbf{e})=m} \hat{\gamma}_{\mathbf{d}\mathbf{e}}P'_{\mathbf{e}}, \qquad \text{with} \qquad \hat{\gamma}_{\mathbf{d}\mathbf{e}} = \alpha_{\mathbf{d}\overline{\mathbf{d}}}\,\hat{\beta}_{\overline{\mathbf{d}}\overline{\mathbf{e}}}\,\alpha_{\overline{\mathbf{e}}\mathbf{e}}. \tag{4.291}$$

The equations (4.278) and (4.279) are equivalent — not only numerically, but also in the sense, the coefficients $\hat{\gamma}_{\mathbf{de}}$ represent an algebra isomorphism if and only if the coefficients $\hat{\beta}_{\mathbf{bc}}$ represent an algebra isomorphism. In order to show this, we first assume $\hat{\pi}$ to be an algebra isomorphism. With equation (4.260) we get for $A, B \in \Lambda_n^-$

$$\begin{aligned}\hat{\rho}(A \vee B) &= \hat{\rho}\big(\hat{\pi}^{-1}\hat{\rho}(A) \vee \hat{\pi}^{-1}\hat{\rho}(B)\big) \\ &= \frac{1}{\lambda^{n-1}\det\hat{\pi}}\hat{\rho}\hat{\pi}^{-1}(\hat{\rho}(A) \wedge \hat{\rho}(B)) \\ &= \mu \cdot \hat{\rho}(A) \wedge \hat{\rho}(B) \quad \text{with} \quad \mu = \frac{1}{\lambda^{n-1}\det\hat{\pi}},\end{aligned} \tag{4.292}$$

i. e., the transformation $\hat{\rho}$ preserves the exterior products and represents an algebra isomorphism from the minus to the plus approach. With equation (4.292) and

$$\begin{aligned}\mathbf{I}^- &= \hat{\pi}^{-1}\hat{\rho}(\mathbf{I}^-) = \mu^{n-1}\det\hat{\rho}\cdot\hat{\pi}^{-1}(\mathbf{I}^-) = \mu^{n-1}\det\hat{\rho}\cdot\hat{\pi}^{-1}(\mathbf{Z}^+) \\ &= \lambda\mu^{n-1}\det\hat{\rho}\cdot\mathbf{Z}^+ = \lambda\mu^{n-1}\det\hat{\rho}\cdot\mathbf{I}^-\end{aligned} \tag{4.293}$$

we get equations (4.281).

Secondly we assume $\hat{\rho}$ to be an algebra isomorphism. With equation (4.263) we get for $A, B \in \Lambda_n^+$

$$\begin{aligned}\hat{\pi}(A \wedge B) &= \hat{\pi}\big(\hat{\rho}^{-1}\hat{\pi}(A) \wedge \hat{\rho}^{-1}\hat{\pi}(B)\big) \\ &= \frac{1}{\mu^{n-1}\det\hat{\rho}}\hat{\pi}\hat{\rho}^{-1}(\hat{\pi}(A) \vee \hat{\pi}(B)) \\ &= \lambda \cdot \hat{\pi}(A) \vee \hat{\pi}(B) \quad \text{with} \quad \lambda = \frac{1}{\mu^{n-1}\det\hat{\rho}},\end{aligned} \tag{4.294}$$

i. e., the transformation $\hat{\pi}$ preserves the exterior products and represents an algebra isomorphism from the plus to the minus approach. With equation (4.294) and

$$\begin{aligned}\mathbf{I}^+ &= \hat{\rho}^{-1}\hat{\pi}(\mathbf{I}^+) = \lambda^{n-1}\det\hat{\pi}\cdot\hat{\rho}^{-1}(\mathbf{I}^+) = \lambda^{n-1}\det\hat{\pi}\cdot\hat{\rho}^{-1}(\mathbf{Z}^-) \\ &= \mu\lambda^{n-1}\det\hat{\pi}\cdot\mathbf{Z}^- = \mu\lambda^{n-1}\det\hat{\pi}\cdot\mathbf{I}^+\end{aligned} \tag{4.295}$$

we get equations (4.281) again. □

Theorem 4.39 is only true for algebra isomorphisms and not for algebra homomorphisms in general, as we will see with the following examples.

Examples. Let $n = 3$. The dimension of the exterior double algebras Λ_3 and Λ_3' then is $\dim\Lambda_3 = \dim\Lambda_3' = 2^3 = 8$.

(E1) We first look at the degenerate homomorphism in the plus approach with $\lambda = 1$

$$\begin{aligned}\pi : \quad \Lambda_3 \quad &\longrightarrow \quad \Lambda_3' \\ P_{\mathbf{b}} \quad &\longmapsto \quad \pi(P_{\mathbf{b}}) = \beta_{\mathbf{bb}}P_{\mathbf{b}}'\end{aligned} \tag{4.296}$$

given by

$$B_{\overline{0}} := (0), \qquad B_{\overline{1}} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.297}$$

Then, the 2- and 3-vectors transform according to the matrices

$$B_{\overline{2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_{\overline{3}} = (0). \tag{4.298}$$

By using equation (4.274) of Theorem 4.38 the matrices

$$\Gamma_{\overline{0}} = (0), \qquad \Gamma_{\overline{1}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.299}$$

$$\Gamma_{\overline{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma_{\overline{3}} = (0) \tag{4.300}$$

are associated to the matrices $B_{\overline{k}}$ from equations (4.297) and (4.298).

Obviously, the matrices $\Gamma_{\overline{k}}$ from equations (4.299) and (4.300) do not represent a (degenerate) algebra homomorphism in the minus approach of the form

$$\begin{array}{cccc} \rho : & \Lambda_3 & \longrightarrow & \Lambda'_3 \\ & E_{\mathbf{b}} & \longmapsto & \rho(E_{\mathbf{b}}) = \gamma_{\mathbf{bb}} E'_{\mathbf{b}} \end{array} \tag{4.301}$$

with e. g. $\mu = 1$.

(E2) Secondly we look at the degenerate homomorphism from the plus to the minus approach with $\lambda = 1$

$$\begin{array}{cccc} \hat{\pi} : & \Lambda_3 & \longrightarrow & \Lambda'_3 \\ & P_{\mathbf{b}} & \longmapsto & \pi(P_{\mathbf{b}}) = \hat{\beta}_{\mathbf{bb}} E'_{\mathbf{b}} \end{array} \tag{4.302}$$

given by

$$\hat{B}_{\overline{0}} := (0), \qquad \hat{B}_{\overline{1}} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.303}$$

Then, the 2- and 3-vectors transform according to the matrices

$$\hat{B}_{\overline{2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{B}_{\overline{3}} = (0). \tag{4.304}$$

By using equation (4.279) of Theorem 4.38 the matrices

$$\hat{\Gamma}_{\overline{0}} = (0), \qquad \hat{\Gamma}_{\overline{1}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.305}$$

$$\hat{\Gamma}_{\overline{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{\Gamma}_{\overline{3}} = (0) \tag{4.306}$$

are associated to the matrices $\hat{B}_{\overline{k}}$ from equations (4.303) and (4.304).

Obviously, the matrices $\hat{\Gamma}_{\overline{k}}$ from equations (4.305) and (4.306) do not represent a (degenerate) algebra homomorphism from the minus to the plus approach of the form

$$\begin{aligned} \hat{\rho}: \quad \Lambda_3 \quad &\longrightarrow \quad \Lambda'_3 \\ E_{\mathbf{b}} \quad &\longmapsto \quad \hat{\rho}(E_{\mathbf{b}}) = \hat{\gamma}_{\mathbf{bb}} P'_{\mathbf{b}} \end{aligned} \tag{4.307}$$

with e. g. $\mu = 1$.

4.6. Chains of Algebra Homomorphisms

Let us shortly focus next on chains of algebra homomorphisms and its concatenation. An example for such a chain is

$$\Lambda_n \quad \xrightarrow{\phi_1} \quad \Lambda'_n \quad \xrightarrow{\phi_2} \quad \Lambda''_n \quad \xrightarrow{\phi_3} \quad \Lambda'''_n. \tag{4.308}$$

In general, a chain can be finite of any length, can be extended to infinitely many steps to just one end or to both ends.

Theorem 4.40 (Associativity). *Let ϕ_i, $i \in \{1,2,3\}$, be three generic algebra homomorphisms according to Notation 4.22 and belonging to the chain (4.308). The concatenation of algebra homomorphisms is associative,*

$$(\phi_3\phi_2)\phi_1 = \phi_3(\phi_2\phi_1). \tag{4.309}$$

Proof. The associativity on the level of the algebra homomorphisms traces back to the associativity of the elements in the field $\mathbb{F}$ with respect to multiplication. Depending on which types of algebra homomorphisms are involved, it may be necessary to switch the basis from the plus to the minus or from the minus to the plus approach before performing the (next) algebra homomorphism. The details are left to the reader. □

Theorem 4.41 (Inverse Algebra Isomorphism). *Let $\phi: \ \Lambda_n \to \Lambda'_n$ denote a generic algebra isomorphism. Then there is one and only one inverse algebra isomorphism $\phi^{-1}: \ \Lambda'_n \to \Lambda_n$ with*

$$\phi\phi^{-1} = \phi'_{\mathrm{Id}}, \qquad \phi^{-1}\phi = \phi_{\mathrm{Id}}. \tag{4.310}$$

Proof. Compare Theorem 4.23. □

4.7. Algebra Endo- and Automorphisms

Let us now specialize to the algebra endomorphisms and automorphisms, i. e. we choose the unity free exterior double algebras in the chain of equation (4.308) to be the same,

$$\Lambda_n = \Lambda'_n = \Lambda''_n = \Lambda'''_n. \tag{4.311}$$

Notation 4.42 (Sets of Algebra Endo- and Automorphisms). We denote with $\mathcal{H}$ the set of all algebra endomorphisms, with $\mathcal{H}_{even}$ the set of all *even* algebra endomorphisms and with $\mathcal{H}_{odd}$ the set of all *odd* algebra endomorphisms:

$$\mathcal{H} := \{\phi \mid \phi \text{ represents an algebra endomorphism}\}\,, \tag{4.312}$$

$$\mathcal{H}_{even} := \{\phi \mid \phi \text{ represents an even algebra endomorphism}\}\,, \tag{4.313}$$

$$\mathcal{H}_{odd} := \{\phi \mid \phi \text{ represents an odd algebra endomorphism}\}\,. \tag{4.314}$$

We denote with $\mathcal{P}$ the set of all algebra automorphisms, with $\mathcal{P}_{even}$ the set of all *even* algebra automorphisms and with $\mathcal{P}_{odd}$ the set of all *odd* algebra automorphisms:

$$\mathcal{P} := \{\phi \mid \phi \text{ represents an algebra automorphism}\}\,, \tag{4.315}$$

$$\mathcal{P}_{even} := \{\phi \mid \phi \text{ represents an even algebra automorphism}\}\,, \tag{4.316}$$

$$\mathcal{P}_{odd} := \{\phi \mid \phi \text{ represents an odd algebra automorphism}\}\,. \tag{4.317}$$

Clearly we have

$$\mathcal{P} \subset \mathcal{H}, \qquad \mathcal{P}_{even} \subset \mathcal{H}_{even}, \qquad \mathcal{P}_{odd} \subset \mathcal{H}_{odd}. \tag{4.318}$$

Theorem 4.43 (Neutral Element). *Let ϕ be any algebra homomorphism and let ϕ_{Id} denote the identity mapping according to Notation 4.22. ϕ_{Id} is the neutral element with respect to concatenation of algebra homomorphisms,*

$$\phi\,\phi_{\mathrm{Id}} = \phi_{\mathrm{Id}}\,\phi = \phi. \tag{4.319}$$

Proof.

$$K_{\phi}\,\mathbb{I} = \mathbb{I}\,K_{\phi} = K_{\phi}. \tag{4.320}$$

□

Theorem 4.44 (Groups of Algebra Automorphisms). *$\mathcal{P}$ with respect to concatenation of its algebra automorphisms represents a group $(\mathcal{P},\ )$. Its neutral element is the algebra automorphism ϕ_{Id}. The inverse element ϕ^{-1} to a generic algebra automorphism ϕ is given by Theorem (4.23).*

$(\mathcal{P}_{even},\ )$ is a subgroup of $(\mathcal{P},\ )$.

Proof. $(\mathcal{P},\ )$ is associative according to Theorem 4.40, has a neutral element ϕ_{Id} according to Theorem 4.43 and provides to any algebra automorphism $\phi \in \mathcal{P}$ an inverse element ϕ^{-1} according to Theorem 4.41.

$(\mathcal{P}_{even},\ )$ is a subgroup of $(\mathcal{P},\ )$, since ϕ_{Id}, with ϕ also ϕ^{-1} and with two even algebra automorphisms ϕ_1 and ϕ_2 also the product $\phi_1\phi_2$ are all even algebra automorphisms. □

5. Projective Geometry. System of Axioms and Basic Concepts

After the discovery of the principle of duality by the French mathematicians Jean Victor Poncelet (1788-1867) and Joseph-Diaz Gergonne (1771-1859) in the first quarter of the nineteenth century, projective geometry experienced an impulse of development for about one century. This impulse expressed itself in a synthetic as well as in an analytic form. In the course of the twentieth century the synthetic approach became less and less known although synthetic projective geometry had led to a wealth of new concepts and an unsurpassed beauty. An example for the new concepts rising out of synthetic projective geometry is the notion of counterspace [Con08, pp. 55].

Nevertheless, the synthetic approach to projective geometry was further developed in the twentieth century by authors like e. g. OSWALD VEBLEN (1880-1960) [VY10, Veb18], GEORGE ADAMS (1894-1963) [Ada65], LOUIS LOCHER (1906-1962) [Loc80b, Loc80a, LE70, Loc70], HAROLD SCOTT MACDONALD COXETER (1907-2003) [Cox74], LAWRENCE EDWARDS (1912-2004) [Edw03], HANNS-JÖRG STOSS [Sto95, Sto99], GERHARD KOWOL [Kow09] and RENATUS ZIEGLER [Zie12].

The purpose of this section is to provide a system of axioms for projective geometry $\mathcal{P}_n$ in terms of the unity free exterior double algebra Λ_n. The aim thereby was to analytically represent into every detail the wealth of synthetic projective geometry as it is described e. g. in the books by LOUIS LOCHER [Loc80b, Loc80a, LE70].

Definition 5.1 (Projective Algebra Λ_n). Since projective geometry $\mathcal{P}_n$ is going to be defined in terms of the unity free exterior double $\mathbb{F}$-algebra $\Lambda_n(+,\cdot,\wedge,\vee)$, we will call the latter from now on shorter as *projective algebra* or *projective* $\mathbb{F}$-*algebra* $\Lambda_n(+,\cdot,\wedge,\vee)$.

5.1. A System of Axioms for Projective Geometry $\mathcal{P}_n$

Definition 5.2 (Equivalence Relation and Equivalence Classes). Two multi vectors A and B of a projective $\mathbb{F}$-algebra Λ_n are called *equivalent*, if and only if their homogeneous parts $\langle A\rangle_k$ and $\langle B\rangle_k$ differ each in a non zero number $\xi_k \in \mathbb{F}\setminus\{0\}$ for all k-vector parts,

$$A \simeq B \quad :\Longleftrightarrow \quad \langle A\rangle_k = \xi_k \langle B\rangle_k \quad \forall k \in \{0,1,\dots,n\}. \tag{5.1}$$

The corresponding equivalence class to a multi vector A is denoted by $[A]$.

The equivalence relation $\simeq$ of Definition 5.2 is reflexive, symmetric and transitive and thus well defined. Reduced to ordinary homogeneous multi vectors, the equivalence relation $\simeq$ represents the well know equivalence relation between homogeneous coordinates in projective geometry.

Definition 5.3 (Projective Geometry $\mathcal{P}_n$). Let $\Lambda_n(+,\cdot,\wedge,\vee)$ be a projective $\mathbb{F}$-algebra. Projective geometry $\mathcal{P}_n$ of dimension 2^n is determined in terms of projective algebra Λ_n by the following axioms:

(A1) *Elements of Projective Geometry.*

- (a) There are $n+1$ different types of *basic elements* corresponding to the $n+1$ different vector subspaces Λ_n^k of projective algebra Λ_n. The basic elements of a certain type (called *k-elements*) are represented by the homogeneous multi vectors $X_{\bar{k}}$ of one of the $n+1$ different vector subspaces Λ_n^k.

(b) A multi vector M of the vector space $\Lambda_n(+,\ )$ represents an *element*, i. e. in general of each type of basic element exactly one,

$$M = \sum_{k=0}^{n} \langle M \rangle_k. \tag{5.2}$$

(c) Equivalent multi vectors represent the same element, i.e. all multi vectors $X \in [A]$ represent the same element as A does.

(A2) *Incidence Relation.* Two elements $[A]$ und $[B]$ are incident if and only if their corresponding homogeneous parts $\langle A \rangle_k$ and $\langle B \rangle_l$ meet the conditions

$$\left.\begin{array}{rcl} \langle A \rangle_k \wedge \langle B \rangle_l & = & 0, \\ \langle A \rangle_k \vee \langle B \rangle_l & = & 0, \end{array}\right\} \quad \forall\ k, l \in \{0, 1, \dots, n\}. \tag{5.3}$$

(A3) *Intersection and Connection.* The geometric operation of connection corresponds to the major exterior product ($\wedge$), the geometric operation of intersection to the minor exterior product ($\vee$).

Remark. There are different conventions of how to interpret the exterior products in geometry. We use — first in the context of projective geometry and then later also in the context of metric geometries — the major exterior product $\wedge$ for the operation of connection and the minor exterior product $\vee$ for the operation of intersection. The same convention can be found in [Zad94, Kapitel XII], [HZ91], [DL03, equation 10.9] and [LD09, Chapter 11]. Examples for the opposite convention are [RGO09, pp. 29-30], [RG11, p. 44] and [Gun11a, p. 18].

Please bear in mind, this only is a different convention and not a fundamental difference. We just need to know, who is using which convention, and then can understand each other.

As a consequence of the transition from projective algebra Λ_n to Clifford double algebra Γ_n in Section 8, the geometric interpretation of the exterior products in the so called homogeneous models of the Calyley-Klein geometries is the same as in projective geometry. This is why we cannot choose a different geometric interpretation for the exterior products in the plane-based geometric algebras of LEO DORST and STEVEN DE KENINCK [DDK24] or the projective geometric algebras of CHARLES GUNN. [Gun17] Compare also Subsection 8.3.

Examples. According to Axiom (A3) of Definition 5.3, the following examples illustrate the geometric operations of connection and intersection, which come along with the major and the minor exterior product respectively. In all examples we use an harmonic system of bases $\{P_{\mathbf{b}}\}$, $\{E_{\mathbf{b}}\}$ according to Definition 4.26.

$\mathcal{P}_2$	**line**	
	$\Lambda_2^{0+} = \Lambda_2^{2-}$	line as a pencil of planes
	$\Lambda_2^{1+} = \Lambda_2^{1-}$	points in the plus approach planes in the minus approach
	$\Lambda_2^{2+} = \Lambda_2^{0-}$	line as a range of points
$\mathcal{P}_2$	**incident point-plane-pair**	
	$\Lambda_2^{0+} = \Lambda_2^{2-}$	point of the incident point-plane-pair
	$\Lambda_2^{1+} = \Lambda_2^{1-}$	lines (of the pencil of lines)
	$\Lambda_2^{2+} = \Lambda_2^{0-}$	plane of the incident point-plane-pair
$\mathcal{P}_3$	**planar field**	
	$\Lambda_3^{0+} = \Lambda_3^{3-}$	field of lines as such
	$\Lambda_3^{1+} = \Lambda_3^{2-}$	points
	$\Lambda_3^{2+} = \Lambda_3^{1-}$	lines
	$\Lambda_3^{3+} = \Lambda_3^{0-}$	field of points as such
$\mathcal{P}_3$	**centric bundle**	
	$\Lambda_3^{0+} = \Lambda_3^{3-}$	bundle of planes as such
	$\Lambda_3^{1+} = \Lambda_3^{2-}$	lines
	$\Lambda_3^{2+} = \Lambda_3^{1-}$	planes
	$\Lambda_3^{3+} = \Lambda_3^{0-}$	bundle of lines as such
$\mathcal{P}_4$	**space**	
	$\Lambda_4^{0+} = \Lambda_4^{4-}$	space of planes as such
	$\Lambda_4^{1+} = \Lambda_4^{3-}$	points
	$\Lambda_4^{2+} = \Lambda_4^{2-}$	linear complexes (including the lines)
	$\Lambda_4^{3+} = \Lambda_4^{1-}$	planes
	$\Lambda_4^{4+} = \Lambda_4^{0-}$	space of points as such

TABLE 4. Basic elements in the projective geometries $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$. In space $\mathcal{P}_4$, the bivectors form a 5-dimensional projective space and represent the linear manifold of the linear complexes. The latter includes the lines l, which satisfy the equivalent line conditions $l \wedge l = \mathbf{0}$ and $l \vee l = \mathbf{0}$. All lines in space form a 4-dimensional quadratic manifold. The basic elements of the centric bundle $\mathcal{P}_3$ are lines and planes. No points are present as basic elements. This is an example, where we can address the lines as *hyperpoints*. In projective geometry $\mathcal{P}_n$ with $n = 2$ the subspaces $\Lambda_n^{1\pm}$ and $\Lambda_n^{(n-1)\mp}$ coincide. In geometry this is possible when the elements of $\Lambda_2^{1\pm}$ represent either lines (in case of the incident point-plane-pair) or in the plus approach points and in the minus approach planes (in case of a single line).

(E1) In projective geometry $\mathcal{P}_4$ of space (Table 4), three points $A = \langle A\rangle_1^+$, $B = \langle B\rangle_1^+$ and $C = \langle C\rangle_1^+$ in general position — compare Definition 5.13 — determine one and exactly one connecting plane

$$A \wedge B \wedge C = \langle A \wedge B \wedge C\rangle_3^+ \neq \mathbf{0}. \tag{5.4}$$

If we take $A = P_{0001}$, $B = P_{0010}$ and $C = P_{0100}$, we get

$$P_{0111} = P_{0001} \wedge P_{0010} \wedge P_{0100}. \tag{5.5}$$

Two lines $l = \langle l\rangle_2^+$ (with $l \wedge l = \mathbf{0}$) and $h = \langle h\rangle_2^+$ (with $h \wedge h = \mathbf{0}$) meet, whenever their connection vanishes,

$$l \wedge h = \mathbf{0}, \tag{5.6}$$

and they are skew, whenever their connection results in the space of all points as such,

$$l \wedge h = \langle l \wedge h\rangle_4^+ \neq \mathbf{0}. \tag{5.7}$$

If we take $l = P_{0011}$ and $h = P_{0110}$, i. e. the two lines meet in P_{0010}, we get

$$l \wedge l = \mathbf{0}, \qquad h \wedge h = \mathbf{0}, \qquad l \wedge h = \mathbf{0}. \tag{5.8}$$

If we take $l = P_{0011}$ and $h = P_{1100}$, i. e. the two lines are skew, we get

$$l \wedge l = \mathbf{0}, \qquad h \wedge h = \mathbf{0}, \qquad l \wedge h = \mathbf{I}^+. \tag{5.9}$$

In order theory, the join of two elements is defined as least upper bound (supremum). In case of two meeting lines in space, the join results in the plane spanned by the two lines. Here, with projective algebra, the result of the operation of connection is the zero vector. Following Definition 5.3 of projective geometry in the case of $n = 4$, there is for two meeting lines in space always exactly one plane, which is incident with both lines. In the example of $l = P_{0011}$ and $h = P_{0110}$, this plane is P_{0111}.

In case only points are involved, the order theoretical operation of join and the operation of connection from projective algebra deliver the same result. Compare as an example equations (5.4) and (5.5).

(E2) In projective geometry $\mathcal{P}_4$ of space (Table 4), three planes $A = \langle A\rangle_1^-$, $B = \langle B\rangle_1^-$ and $C = \langle C\rangle_1^-$ in general position — compare Definition 5.13 — determine one and exactly one intersecting point

$$A \vee B \vee C = \langle A \vee B \vee C\rangle_3^- \neq \mathbf{0}. \tag{5.10}$$

If we take $A = E_{0001}$, $B = E_{0010}$ and $C = E_{0100}$, we get

$$E_{0111} = E_{0001} \vee E_{0010} \vee E_{0100}. \tag{5.11}$$

Two lines $l = \langle l\rangle_2^-$ (with $l \vee l = \mathbf{0}$) and $h = \langle h\rangle_2^-$ (with $h \vee h = \mathbf{0}$) are co-planar, whenever their intersection vanishes,

$$l \vee h = \mathbf{0}, \tag{5.12}$$

and they are skew, whenever their intersection results in the space of all planes as such,

$$l \vee h = \langle l \vee h \rangle_4^- \neq \mathbf{0}. \tag{5.13}$$

If we take $l = E_{0011}$ and $h = E_{0110}$, i. e. the two lines share the common plane E_{0010}, we get

$$l \vee l = \mathbf{0}, \qquad h \vee h = \mathbf{0}, \qquad l \vee h = \mathbf{0}. \tag{5.14}$$

If we take $l = E_{0011}$ and $h = E_{1100}$, i. e. the two lines are skew, we get

$$l \vee l = \mathbf{0}, \qquad h \vee h = \mathbf{0}, \qquad l \vee h = \mathbf{I}^-. \tag{5.15}$$

In order theory, the meet of two elements is defined as greatest lower bound (infimum). In case of two co-planar lines in space, the meet results in the point shared by the two lines. Here, with projective algebra, the result of the operation of intersection is the zero vector. Following Definition 5.3 of projective geometry in the case of $n = 4$, there is for two co-planar lines in space always exactly one point, which is incident with both lines. In the example of $l = E_{0011}$ and $h = E_{0110}$, this point is E_{0111}.

In case only planes are involved, the order theoretical operation of meet and the operation of intersection from projective algebra deliver the same result. Compare equations (5.10) and (5.11).

(E3) In projective geometry $\mathcal{P}_3$ of the planar field (Table 4), two different lines $A = \langle A \rangle_1^-$ and $B = \langle B \rangle_1^-$ determine one and exactly one intersecting point

$$A \vee B = \langle A \vee B \rangle_2^- \neq \mathbf{0}. \tag{5.16}$$

If we take $A = E_{001}$ and $B = E_{010}$, we get

$$E_{011} = E_{001} \vee E_{010}. \tag{5.17}$$

In this lower dimensional case, the order theoretical operation of meet and the operation of intersection from projective algebra deliver for two different lines the same result, i. e. the meeting point.

(E4) In projective geometry $\mathcal{P}_3$ of the centric bundle (Table 4), two different lines $A = \langle A \rangle_1^+$ and $B = \langle B \rangle_1^+$ determine one and exactly one connecting plane

$$A \wedge B = \langle A \wedge B \rangle_2^+ \neq \mathbf{0}. \tag{5.18}$$

If we take $A = P_{001}$ and $B = P_{010}$, we get

$$P_{011} = P_{001} \wedge P_{010}. \tag{5.19}$$

In this lower dimensional case, the order theoretical operation of join and the operation of intersection from projective algebra deliver for two different lines the same result, i. e. the common plane.

Remark. Let us emphasise the difference of how usually *join* and *meet* are understood and how in terms of projective algebra the operations of connection and intersection are defined. In order theory we have:

- The join of two elements equals the least upper bound (*supremum*).
- The meet of two elements equals the greatest lower bound (*infimum*).

This usually is also applied in projective geometry.

In this article, we define the operations of connection and intersection according to Definition 5.3 in terms of the major and minor exterior products respectively. As the above displayed examples show, the definitions of join and meet in the context of order theory and the definition of the operations of connection and intersection in the context of projective algebra do not always lead to the same results. This is also why we are not using the terms 'join' and 'meet' but instead 'operation of connection' and 'operation of intersection' or just 'connect' and 'intersect' respectively.

What is the advantage of defining the operations of connection and intersection in terms of the major and minor exterior products respectively? As we will see later on, it leads to incidence relations, which are in space valid not only for points and planes, but also for lines and linear complexes.

So our attitude towards the geometric operations of connection and intersection is a learning one. We let us teach from the exterior products, what the meaning of connection and intersection is — especially in the projective space of all lines and in the projective space of all linear complexes.

Definition 5.4 (Real, Complex or Finite Projective Geometry). Depending on the field $\mathbb{F}$ of the projective $\mathbb{F}$-algbera Λ_n we have a) for $\mathbb{F} = \mathbb{R}$ real projective geometry, b) for $\mathbb{F} = \mathbb{C}$ complex projective geometry and c) for finite $\mathbb{F}$ finite projective geometry.

Definition 5.5 (Names of the Basic Elements for $n \in \{2, 3, 4\}$). In case of the projective geometries $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ we use the in Table 4 listed names for the different types of basic elements.

Definition 5.6 (Space and Counterspace. Projective Version). Inasmuch as projective geometry $\mathcal{P}_n$ is expressed in terms of the plus approach Λ_n^+ it is called *space* or *projective space* and inasmuch as it is expressed in terms of the minus approach Λ_n^- it is called *counterspace* or *projective counterspace.*

Axiom (A2) of Definition 5.3 shows one way of how incidence can be determined. According to this Axiom, incidence is a symmetric relation and, in general, not reflexive, since there are 2-vectors $C_{\overline{2}}$ in Λ_4 with $C_{\overline{2}} \Diamond C_{\overline{2}} \neq \mathbf{0}$.

Projective algebra Λ_n is with respect to both exterior products $\lozenge$ non commutative. This is why it would be natural to require all commutations of two elements $[A]$ and $[B]$ to vanish, whenever they coincide. The following theorem shows three equivalent assertions to determine incidence, one of which is the before mentioned assumption.

Theorem 5.7. *For two multi vectors $A, B \in \Lambda_n$ we denote the equivalent multi vectors by $X \in [A]$ and $Y \in [B]$. The following assertions are equivalent:*

(a) $\langle A\rangle_k \lozenge \langle B\rangle_l = \mathbf{0} \quad \forall\, k, l \in \{0, 1, \dots, n\}$,
(b) $\{X, Y\}_{\lozenge} = \mathbf{0}$ *and* $[X, Y]_{\lozenge} = \mathbf{0} \quad \forall\, X \in [A]$ *and* $\forall\, Y \in [B]$,[12]
(c) $\{Y, X\}_{\lozenge} = \mathbf{0}$ *and* $[Y, X]_{\lozenge} = \mathbf{0} \quad \forall\, X \in [A]$ *and* $\forall\, Y \in [B]$,
(d) $X \lozenge Y = Y \lozenge X = \mathbf{0} \quad \forall\, X \in [A]$ *and* $\forall\, Y \in [B]$.

Proof. With $X = \sum_{k=0}^{n} \nu_k \langle A\rangle_k\ \forall\, \nu_k \in \mathbb{F}\setminus\{0\}$ and with $Y = \sum_{l=0}^{n} \xi_l \langle B\rangle_l\ \forall\, \xi_l \in \mathbb{F}\setminus\{0\}$ we get all $X \in [A]$ and all $Y \in [B]$.

$$(a) \Longleftrightarrow \left\{ \begin{array}{lcll} X \lozenge Y - Y \lozenge X & = & \mathbf{0}, & \forall\, X \in [A] \\ X \lozenge Y + Y \lozenge X & = & \mathbf{0}, & \forall\, Y \in [B] \end{array} \right\} \Longleftrightarrow (b).$$

$(b) \Longleftrightarrow (c)$ is left to the reader.

With $X \lozenge Y = \dfrac{1}{2}(2\,X \lozenge Y + Y \lozenge X - Y \lozenge X) = \dfrac{1}{2}[X, Y]_{\lozenge} + \dfrac{1}{2}\{X, Y\}_{\lozenge}$
and $Y \lozenge X = \dfrac{1}{2}[Y, X]_{\lozenge} + \dfrac{1}{2}\{Y, X\}_{\lozenge} = -\dfrac{1}{2}[X, Y]_{\lozenge} + \dfrac{1}{2}\{X, Y\}_{\lozenge}$
we get : $(b) \Longleftrightarrow (d)$.

□

Examples. With Axiom (A2) of Definition 5.3 and specifically the conditions of equations (5.3), incidence is defined in projective geometry $\mathcal{P}_n$. The following examples illustrate the incidence of two geometric elements A and B. We again use an harmonic system of bases $\{P_{\mathbf{b}}\}$, $\{E_{\mathbf{b}}\}$ according to Definition 4.26.

(E1) In space $\mathcal{P}_4$, determine all points $A = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}}$ lying in the line $B = -5P_{1001}$. Since in the minus approach any point $A = \langle A\rangle_3^-$ is of grade 3 and the line $B = \langle B\rangle_2^-$ of grade 2, the product $A \vee B = \mathbf{0}$ vanishes trivially – compare equation (4.28) – and gives us no condition for the incidence of the points A and the line B. Thus, the two conditions of equation (5.3) reduce to one relevant condition,

$$\begin{aligned} \mathbf{0} &= A \wedge B \qquad (5.20) \\ &= \left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}} \right) \wedge (-5P_{1001}) \end{aligned}$$

---

[12] We are adopting the often used bracket notation for the commutator, $[S, T]_{\times} := S \times T - T \times S$, and the anti commutator, $\{S, T\}_{\times} := S \times T + T \times S$. × denotes the product and can be omitted as a subscript of the commutators, if it is clear from the context, which product has to be used.

$$
\begin{aligned}
&= (\lambda_{0010}P_{0010} + \lambda_{0100}P_{0100}) \wedge (-5P_{1001}) \\
\Longleftrightarrow \quad & \lambda_{0010} = 0, \quad \lambda_{0100} = 0 \\
\Longleftrightarrow \quad & A = \lambda_{0001}P_{0001} + \lambda_{1000}P_{1000} \\
& \forall\ \lambda_{0001}, \lambda_{1000} \in \mathbb{F} \quad \text{and} \quad (\lambda_{0001}, \lambda_{1000}) \neq (0,0)
\end{aligned}
$$

(E2) In the planar field $\mathcal{P}_3$, determine all points $A = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}}$ lying in the line $B = P_{101} - P_{011}$. With Theorem 4.27 the two incidence conditions (5.3) are equivalent, i. e. there is one relevant equation left,

$$
\mathbf{0} = A \wedge B = \left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}} \right) \wedge (P_{101} - P_{011}) \tag{5.21}
$$

$$
\begin{aligned}
&= -\lambda_{010}\mathbf{I}^{+} - \lambda_{100}\mathbf{I}^{+} = -(\lambda_{010} + \lambda_{100})\mathbf{I}^{+} \\
\Longleftrightarrow \quad & \lambda_{010} = -\lambda_{100} \\
\Longleftrightarrow \quad & A = \lambda_{001}P_{001} - \lambda_{100}P_{010} + \lambda_{100}P_{100} = \\
& \quad = \lambda_{001}P_{001} + \lambda_{100}(-P_{010} + P_{100}) \\
& \forall\ \lambda_{001}, \lambda_{100} \in \mathbb{F} \quad \text{and} \quad (\lambda_{001}, \lambda_{100}) \neq (0,0)
\end{aligned}
$$

(E3) In projective geometry $\mathcal{P}_n$ of arbitrary dimension n, determine all points (hyperpoints) $A = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}}$ lying in the projective space of points (hyperpoints) as such $\mathbf{I}^{+}$. In projective algebra Λ_n the two incidence conditions of equation (5.3) hold trivially, i. e., all points (hyperpoints) $A = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}}$ are part of the projective space of points (hyperpoints) as such $\mathbf{I}^{+}$.

In the usual (double) Graßmann algebra, the minor exterior product would not vanish, i. e., we would have $A \wedge \mathbf{I}^{+} = \mathbf{0}$ and $A \vee \mathbf{I}^{+} = A \vee \mathbf{Z}^{-} = A$. Thus, the equations (5.3) would not in general define incidence.

(E4) In space $\mathcal{P}_4$, we are looking for all lines $A = \sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} P_{\mathbf{b}}$, which coincide with the incident point-plane-pair $B = P_{0001} + P_{1101}$.

A is representing a line if and only if $A \between A = \mathbf{0}$. This is the line or Plücker condition for a linear complex A to represent a line, which is also called a special linear complex. With Theorem (4.27) it is enough to require

$$
\mathbf{0} = A^{+} \wedge A^{-} = A^{+} \wedge A^{+} = \sum_{\substack{S(\mathbf{b})=2 \\ S(\mathbf{c})=2}} \lambda_{\mathbf{b}} \lambda_{\mathbf{c}} (P_{\mathbf{b}} \wedge P_{\mathbf{c}}) \tag{5.22}
$$

$$
\begin{aligned}
&= \sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} \lambda_{\overline{\mathbf{b}}} \alpha_{\mathbf{b}\overline{\mathbf{b}}} P_{1111} \\
&= 2(\lambda_{0011}\lambda_{1100} - \lambda_{0101}\lambda_{1010} + \lambda_{0110}\lambda_{1001})\mathbf{I}^{+}
\end{aligned}
$$

$$
\Longleftrightarrow \quad \lambda_{0011}\lambda_{1100} - \lambda_{0101}\lambda_{1010} + \lambda_{0110}\lambda_{1001} = 0. \tag{5.23}
$$

Equation (5.23) is the above mentioned line condition in coordinate form.

The point P_{0001} and the plane P_{1101} of the point-plane-pair B are incident, since we have

$$P_{0001} \wedge P_{1101} = \mathbf{0}, \tag{5.24}$$

$$P_{0001} \vee P_{1101} = \alpha_{1110\,0001}\alpha_{0010\,1101} \cdot E_{1110} \vee E_{0010} = \mathbf{0}. \tag{5.25}$$

The incidence conditions (5.3) are

$$\mathbf{0} = A \wedge P_{0001} = \sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} P_{\mathbf{b}} \wedge P_{0001} = \sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} P_{0001} \wedge P_{\mathbf{b}} \tag{5.26}$$

$$\begin{aligned}
&= \lambda_{0110}P_{0111} + \lambda_{1010}P_{1011} + \lambda_{1100}P_{1101} \\
&\iff \quad \lambda_{0110} = 0, \lambda_{1010} = 0, \lambda_{1100} = 0 \\
&\iff \quad A = \lambda_{0011}P_{0011} + \lambda_{0101}P_{0101} + \lambda_{1001}P_{1001} \\
&\qquad\quad \forall\ \lambda_{0011}, \lambda_{0101}, \lambda_{1001} \in \mathbb{F} \\
&\iff \quad A \text{ represents any line from the bundle of lines} \\
&\qquad\quad \text{through } P_{0001}.
\end{aligned}$$

$$\mathbf{0} = A \wedge P_{1101} \quad \forall A \quad \text{(no condition on line } A\text{)} \tag{5.27}$$

$$\mathbf{0} = A \vee P_{0001} \quad \forall A \quad \text{(no condition on line } A\text{)} \tag{5.28}$$

$$\mathbf{0} = A \vee P_{1101} \tag{5.29}$$

$$\begin{aligned}
&= \sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}} \vee (\alpha_{0010\,1101}E_{0010}) \\
&= \alpha_{0010\,1101} \sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}} \vee E_{0010} \\
&= \alpha_{0010\,1101}\,(\lambda_{1010}\alpha_{0101\,1010}E_{0111} + \lambda_{0110}\alpha_{0110\,1001}E_{1011} + \\
&\quad + \lambda_{0011}\alpha_{1100\,0011}E_{1110}) \\
&\iff \quad \lambda_{0011} = 0, \lambda_{0110} = 0, \lambda_{1010} = 0 \\
&\iff \quad A = \lambda_{0101}P_{0101} + \lambda_{1001}P_{1001} + \lambda_{1100}P_{1100} \\
&\qquad\quad \forall \quad \lambda_{0101}, \lambda_{1001}, \lambda_{1001} \in \mathbb{F} \\
&\iff \quad A \text{ represents any line from the field of lines} \\
&\qquad\quad \text{in } P_{1101}.
\end{aligned}$$

Equation (5.26) and (5.29) restrict the solution to

$$A = \lambda_{0101}P_{0101} + \lambda_{1001}P_{1001}, \qquad \forall\ \lambda_{0101}, \lambda_{1001} \in \mathbb{F}, \tag{5.30}$$

i. e., A represents any line of the pencil of lines through the point P_{0001} and in the plane P_{1101} of the incident point-plane-pair B. The solution of equation (5.30) meets the line condition (5.23).

This is an example where two incidence conditions of equations (5.3) are non trivial.

(E5) Are all basic elements of projective geometry $\mathcal{P}_4$ in space incident with itself? Or is the incidence relation reflexive for all basic elements of projective geometry $\mathcal{P}_4$?

$$\text{space of planes as such } \Lambda_4^{0+} \begin{cases} X_{\overline{0}}^{+} \wedge X_{\overline{0}}^{+} = \mathbf{0}, \quad X_{\overline{0}}^{+} \vee X_{\overline{0}}^{+} = \mathbf{0}, \\ \forall\ X_{\overline{0}}^{+} \in \Lambda_4^{0+} \end{cases} \tag{5.31}$$

$$\text{points } \Lambda_4^{1+} \begin{cases} X_{\overline{1}}^{+} \wedge X_{\overline{1}}^{+} = \mathbf{0}, \quad X_{\overline{1}}^{+} \vee X_{\overline{1}}^{+} = \mathbf{0}, \\ \forall\ X_{\overline{1}}^{+} \in \Lambda_4^{1+} \end{cases} \tag{5.32}$$

$$\text{linear complexes } \Lambda_4^{2+} \begin{cases} X_{\overline{2}}^{+} \wedge X_{\overline{2}}^{+} = \mathbf{0}, \quad X_{\overline{2}}^{+} \vee X_{\overline{2}}^{+} = \mathbf{0}, \\ \text{If this condition is satisfied,} \\ \text{the linear complex } X_{\overline{1}}^{+} \in \Lambda_4^{1+} \\ \text{represents a special one,} \\ \text{i.e. a line.} \\ \text{All other linear complexes} \\ X_{\overline{1}}^{+} \in \Lambda_4^{1+} \text{ are not incident} \\ \text{with itself anymore.} \end{cases} \tag{5.33}$$

$$\text{planes } \Lambda_4^{3+} \begin{cases} X_{\overline{3}}^{+} \wedge X_{\overline{3}}^{+} = \mathbf{0}, \quad X_{\overline{3}}^{+} \vee X_{\overline{3}}^{+} = \mathbf{0}, \\ \forall\ X_{\overline{3}}^{+} \in \Lambda_4^{3+} \end{cases} \tag{5.34}$$

$$\text{space of points as such } \Lambda_4^{4+} \begin{cases} X_{\overline{4}}^{+} \wedge X_{\overline{4}}^{+} = \mathbf{0}, \quad X_{\overline{4}}^{+} \vee X_{\overline{4}}^{+} = \mathbf{0}, \\ \forall\ X_{\overline{4}}^{+} \in \Lambda_4^{4+} \end{cases} \tag{5.35}$$

We see, all basic elements of space are incident with itself, except for the non-special linear complexes. An example for such a complex is $X_{\overline{2}}^{+} = P_{0011} + P_{1100}$, since

$$X_{\overline{2}}^{+} \wedge X_{\overline{2}}^{+} = (P_{0011} + P_{1100}) \wedge (P_{0011} + P_{1100}) = 2\mathbf{I}^{+}. \tag{5.36}$$

(E6) We continue with projective geometry $\mathcal{P}_4$ in space. For two basic elements A, B of the same type, when are they incident?

$$\text{two points } A, B \in \Lambda_4^{1+} \begin{cases} A \wedge B = \mathbf{0}, \quad A \vee B = \mathbf{0}, \\ \iff A \simeq B, \text{ i.e. when } A \text{ and } B \\ \text{represent the same point.} \end{cases} \tag{5.37}$$

$$\begin{matrix}\text{two linear complexes} \\ A, B \in \Lambda_4^{2+}\end{matrix} \begin{cases} A \wedge B = \mathbf{0}, \quad A \vee B = \mathbf{0}, \\ \text{(a) } A \wedge A = \mathbf{0} \text{ and } B \wedge B = \mathbf{0} \\ \iff \text{The lines } A \text{ and } B \text{ meet} \\ \text{and are co-planar.} \\ \text{(b) } A \wedge A \neq \mathbf{0} \text{ and } B \wedge B = \mathbf{0} \\ \iff \text{The line } B \text{ belongs to the} \\ \text{linear complex } A. \\ \text{(c) } A \wedge A \neq \mathbf{0} \text{ and } B \wedge B \neq \mathbf{0} \\ \iff \text{The linear complexes } A \text{ and} \\ B \text{ are null-invariant.} \end{cases} \tag{5.38}$$

$$\text{two planes } A, B \in \Lambda_4^{3+} \begin{cases} A \wedge B = \mathbf{0}, \quad A \vee B = \mathbf{0}, \\ \iff A \simeq B, \text{ i.e. when } A \text{ and } B \\ \text{represent the same plane.} \end{cases} \tag{5.39}$$

The case $A \wedge A = \mathbf{0}$ and $B \wedge B \neq \mathbf{0}$ is covered by the case (b) of the incidence equations (5.38), since the the latter are symmetric with respect to A and B.

The last two examples demonstrate the incidence relation in projective geometry $\mathcal{P}_4$ of space. The incidence conditions of Axiom (A2) of Definition (5.3) bring together in case of projective geometry $\mathcal{P}_4$ of space, the incidence relations between points and planes respectively, as we are used to see them, with the incidence relations for the linear complexes. The latter are also know as null-invariance. Compare Definition 3.20 of [Sto99, p. 106]. Limited to the lines of the projective space of linear complexes, the null-invariance becomes the meeting relation (in German: *Treffrelation*) of projective line geometry. Compare Chapter 7 of [Sto95, p. 106].

Remark. The advantage to define incidence for two generic geometric elements A and B according to Axiom (A2) of Definition (5.3) becomes obvious, when proceeding to the incidence relations of spacial projective geometry $\mathcal{P}_4$: This definition integrates the incidence relations for points and planes — which can also be expressed in terms of the order theoretical operators of join and meet — as well as the incidence relations for linear complexes, i. e. null-invariance. [Sto99, p. 106] One and the same definition, of what incidence is, holds for all involved basic elements.

5.2. Principle of Duality

Definition 5.8 (Major Dual). The *major dual* S' of any expression S from projective geometry $\mathcal{P}_n$ such as e. g. an equation, a theorem, or a definition is obtained by interchanging $\wedge$ with $\vee$ and by reversing the sign of the plus-minus notation.

Theorem 5.9 (Major Principle of Duality). *Any statement S from projective geometry $\mathcal{P}_n$ is true if and only if the major dual statement S' from projective geometry $\mathcal{P}_n$ is true.*

Proof. Definition 5.3 of projective geometry $\mathcal{P}_n$ as well as Definition 4.2 of projective $\mathbb{F}$-algebra are symmetric with respect to the plus and minus approach. The major principle of duality is rooted in the axioms of projective geometry $\mathcal{P}_n$. □

Definition 5.10 (Minor Dual). Let S be any expression such as e. g. an equation, a theorem, or a definition in the plus (or minus) approach. of $\mathcal{P}_n$. The

Space Λ_n^+		Counterspace Λ_n^-
$X_{\overline{k}} := \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \mathbf{P_b}$	$\leftrightarrow$	$X'_{\overline{k}} := \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \mathbf{E_b}$
$A := \sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{P_b}$	$\leftrightarrow$	$A' := \sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{E_b}$
$\wedge$	$\leftrightarrow$	$\vee$
$\vee$	$\leftrightarrow$	$\wedge$
$[\sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{P_b}]$ and $[\sum_{\mathbf{c}} \mu_{\mathbf{c}} \mathbf{P_c}]$ are incident.	$\leftrightarrow$	$[\sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{E_b}]$ and $[\sum_{\mathbf{c}} \mu_{\mathbf{c}} \mathbf{E_c}]$ are incident.

TABLE 5. Major principle of duality in projective geometry $\mathcal{P}_n$.

minor dual S'' of S is obtained by translating the major dual S' from the minus approach (or plus approach respectively) to the plus approach (or minus approach respectively).

Theorem 5.11 (Minor Principle of Duality). *Any statement S from projective geometry $\mathcal{P}_n$ is true if and only if the minor dual statement S'' from projective geometry $\mathcal{P}_n$ is true.*

Proof. $S \iff S' \iff S''$. See Table 6. $\square$

Space Λ_n^+		Space Λ_n^+
$X_{\overline{k}} := \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \mathbf{P_b}$	$\leftrightarrow$	$\langle X'_{\overline{k}} \rangle^{+}_{\overline{n-k}} := \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \langle \mathbf{E_b} \rangle^{+}$
$A := \sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{P_b}$	$\leftrightarrow$	$\langle A' \rangle^{+} := \sum_{\mathbf{b}} \lambda_{\mathbf{b}} \langle \mathbf{E_b} \rangle^{+}_{\overline{n-S(\mathbf{b})}}$
$\wedge$	$\leftrightarrow$	$\vee$
$\vee$	$\leftrightarrow$	$\wedge$
$[\sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{P_b}]$ and $[\sum_{\mathbf{c}} \mu_{\mathbf{c}} \mathbf{P_c}]$ are incident.	$\leftrightarrow$	$\langle [\sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{E_b}] \rangle^{+}$ and $\langle [\sum_{\mathbf{c}} \mu_{\mathbf{c}} \mathbf{E_c}] \rangle^{+}$ are incident.

Counterspace Λ_n^-		Counterspace Λ_n^-
$\langle X_{\overline{k}} \rangle^{-}_{\overline{n-k}} := \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \langle \mathbf{P_b} \rangle^{-}$	$\leftrightarrow$	$X'_{\overline{k}} := \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \mathbf{E_b}$
$\langle A \rangle^{-} := \sum_{\mathbf{b}} \lambda_{\mathbf{b}} \langle \mathbf{P_b} \rangle^{-}_{\overline{n-S(\mathbf{b})}}$	$\leftrightarrow$	$A' := \sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{E_b}$
$\wedge$	$\leftrightarrow$	$\vee$
$\vee$	$\leftrightarrow$	$\wedge$
$\langle [\sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{P_b}] \rangle^{-}$ and $\langle [\sum_{\mathbf{c}} \mu_{\mathbf{c}} \mathbf{P_c}] \rangle^{-}$ are incident.	$\leftrightarrow$	$[\sum_{\mathbf{b}} \lambda_{\mathbf{b}} \mathbf{E_b}]$ and $[\sum_{\mathbf{c}} \mu_{\mathbf{c}} \mathbf{E_c}]$ are incident.

TABLE 6. Minor principle of duality in projective geometry $\mathcal{P}_n$.

$n \in \{1,2\}$		Space Λ_n^+	Counterspace Λ_n^-
$\mathcal{P}_2$	**line**		
$k=0$	$m=0$	line	line
$k=1$	$m=0$	point or plane	plane or point
	$m=1$	pencil of points or of planes	pencil of planes or of points
$k=2$	$m=0$	line	line
$\mathcal{P}_2$	**incident point-plane-pair**		
$k=0$	$m=0$	incident point-plane-pair	incident point-plane-pair
$k=1$	$m=0$	line	line
	$m=1$	pencil of lines	pencil of lines
$k=2$	$m=0$	incident point-plane-pair	incident point-plane-pair
$\mathcal{P}_3$	**planar field**		
$k=0$	$m=0$	field of lines as such	field of points as such
$k=1$	$m=0$	point	line
	$m=1$	pencil of points	pencil of lines
	$m=2$	field of points	field of lines
$k=2$	$m=0$	line	point
	$m=1$	pencil of lines	pencil of points
	$m=2$	field of lines	field of points
$k=3$	$m=0$	field of points as such	field of lines as such
$\mathcal{P}_3$	**centric bundle**		
$k=0$	$m=0$	bundle of planes as such	bundle of lines as such
$k=1$	$m=0$	line	plane
	$m=1$	pencil of lines	pencil of planes
	$m=2$	bundle of lines	bundle of planes
$k=2$	$m=0$	plane	line
	$m=1$	pencil of planes	pencil of lines
	$m=2$	bundle of planes	bundle of lines
$k=3$	$m=0$	bundle of lines as such	bundle of planes as such

TABLE 7. k-primitive geometric forms of grade m in $\mathcal{P}_2$ and $\mathcal{P}_3$. The basic elements of a k-primitive geometric form are homogeneous multi vectors of grade k.

5.3. Primitive Geometric Forms

Primitive geometric forms are linear geometric objects. The Tables 7 and 8 list all primitive geometric forms of the projective geometries $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$.

Definition 5.12 (k-primitive Geometric Form of Grade m). Let $X_i = \langle X_i \rangle_k$ denote $m+1$ linear independent k-vectors in projective geometry $\mathcal{P}_n$ with $1 \leq i \leq m+1 \leq \binom{n}{k}$ and $0 \leq k \leq n$. Then the *k-primitive geometric form of grade m* is the sub vector space

$$U := \left\{ \sum_{i=1}^{m+1} \xi_i X_i \,\middle|\, \xi_i \in \mathbb{F} \right\} \tag{5.40}$$

of projective algebra Λ_n with $\dim U = m+1$.

		Space Λ_4^+	Counterspace Λ_4^-
$\mathcal{P}_4$	**space**		
$k=0$	$m=0$	space of planes as such	space of points as such
$k=1$	$m=0$	point	plane
	$m=1$	pencil of points	pencil of planes
	$m=2$	field of points	bundle of planes
	$m=3$	space of points	space of planes
$k=2$	$m=0$	complex	wood
	$m=1$	pencil of complexes	pencil of woods
	$m=2$	bundle of complexes	bundle of woods
	$m=3$	3-manifold of complexes	3-manifold of woods
	$m=4$	4-manifold of complexes	4-manifold of woods
	$m=5$	5-manifold of complexes	5-manifold of woods
$k=3$	$m=0$	plane	point
	$m=1$	pencil of planes	pencil of points
	$m=2$	bundle of planes	field of points
	$m=3$	space of planes	spaces of points
$k=4$	$m=0$	space of points as such	space of planes as such

TABLE 8. k-primitive geometric forms of grade m in $\mathcal{P}_4$. The basic elements of a k-primitive geometric form are homogeneous multi vectors of grade k.

The *general position* of l basic elements of a k-primitive geometric form of grade m is an important conditions in many theorems of projective geometry $\mathcal{P}_n$, e. g. in the Fundamental Theorem of Projective Geometry 6.6 later on.

Definition 5.13 (General Position of l Basic Elements). Let $\mathcal{L}_l := \{i \mid i, l \in \mathbb{N}, 1 \leq i \leq l\}$ be the set of the l first natural numbers. l basic elements X_i, $i \in \mathcal{L}_l$, of a k-primitive geometric form of grade m are said to be in general position

(a) in case of $l \leq m+1$ if and only if the l k-vectors X_i are linearly independent.

(b) in case of $l > m+1$ if and only if for any variation $\{i_1, \ldots, i_{m+1}\}$ of length $m+1$ without repetitions

$$\{i_1, \ldots, i_{m+1}\} \subset \mathcal{L}_l \qquad \text{and} \qquad 1 \leq i_1 \leq \ldots \leq i_{m+1} \leq l \tag{5.41}$$

the k-vectors $X_{i_1}, \ldots, X_{i_{m+1}}$ are linearly independent.

If we have more than $m+1$ basic elements of a k-primitive geometric form of grade m in general position, then it is always possible to choose $m+1$ basic elements and display the rest as linear combinations of the first $m+1$ freely chosen basic elements. This is the content of Theorem 5.14. It will be used later on in the proof of Theorem 6.6.

Theorem 5.14. *Let* $\mathcal{L}_l := \{i \mid i, l \in \mathbb{N}, 1 \le i \le l\}$ *be the set of the* l *first natural numbers and let the* l *basic elements*

$$X_i, \qquad\qquad i \in \mathcal{L}_l, \tag{5.42}$$

of a k*-primitive geometric form of grade* m *be in general position. Further we choose*

$$l > m + 1, \qquad k, m \in \mathbb{N}, \tag{5.43}$$

$$\mathcal{I} := \{i_1, i_2, \ldots, i_{m+1}\}, \qquad 1 \le i_1 < i_2 < \cdots < i_{m+1} \le l, \tag{5.44}$$

$$\overline{\mathcal{I}} = \{i_{m+2}, i_{m+3}, \ldots, i_l\} := \mathcal{L}_l \setminus \mathcal{I}, \tag{5.45}$$

$$1 \le i_{m+2} < i_{m+3} < \ldots < i_l \le l. \tag{5.46}$$

For any set of indices $\mathcal{I}$ *and for any index* $r \in \overline{\mathcal{I}}$ *the coefficients* λ_{rs} *of the linear combination*

$$X_r = \sum_{s \in \mathcal{I}} \lambda_{rs} X_s, \qquad r \in \overline{\mathcal{I}}, \tag{5.47}$$

do not vanish,

$$\lambda_{rs} \neq 0, \qquad\qquad \forall\, \mathcal{I}, \quad \forall\, r \in \overline{\mathcal{I}}, \quad \forall\, s \in \mathcal{I}. \tag{5.48}$$

Proof. By precondition, the basic elements

$$X_s, \qquad\qquad s \in \mathcal{I}, \tag{5.49}$$

are linearly independent for any set of indices $\mathcal{I}$. Thus, with the linear combination of equation (5.47) we can describe all basic elements of the given k-primitive geometric form of grade m and especially the basic element X_r with $r \in \overline{\mathcal{I}}$.

We will proof the assertion by contradiction an thus assume that there is for a certain choice of $\mathcal{I} = \mathcal{I}_1$, $r = r_1$ and $s = s_1$ a vanishing coefficient $\lambda_{r_1 s_1} = 0$. Then there is the set of indices

$$\mathcal{I}_2 := (\mathcal{I}_1 \setminus \{s_1\}) \cup \{r_1\}, \tag{5.50}$$

where the $m + 1$ basic elements X_j, $j \in \mathcal{I}_2$ are linearly dependent. This is in contradiction to the precondition that the l basic elements X_i, $i \in \mathcal{L}_l$ are in general position. □

5.4. Cross Ratio

The cross ratio of four different basic elements is another basic and fundamental concept of projektive geometry $\mathcal{P}_n$. We define it here, look at some of its properties and show its invariance under the operations of connection $\wedge$ or intersection $\vee$.

Definition 5.15 (Cross Ratio). Four different basic elements

$$A = \langle A \rangle_k, \qquad B = \langle B \rangle_k, \qquad C = \langle C \rangle_k, \qquad D = \langle D \rangle_k, \tag{5.51}$$

of a k-primitive geometric form with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B \tag{5.52}$$

form the cross ratio

$$CR(AB\,CD) := \frac{\lambda}{\mu}. \tag{5.53}$$

With respect to the cross ratio, the basic elements A and B are called *base elements*, the basic elements C and D *dividing elements.*

Please note the order of A, B, C, D in the cross ratio. The base element A is the first element in the notation $CR(AB\,CD)$ and in the linear combination of the equations (5.52) it is the basic element, whose coefficient has to be 1. So the order of the base elements in the notation $CR(AB\,CD)$ has to be taken into account. The same is the case for the dividing elements. The first dividing element C in the notation $CR(AB\,CD)$ delivers the coefficient λ for the numerator, the second dividing element D in the notation $CR(AB\,CD)$ delivers the coefficient μ for the denominator of the cross ratio.

In order to show, that the cross ratio is well defined and does not depend on the weight factors of the basic elements A, B, C and D, we replace the latter by

$$A = \alpha' A' \in [A],\ B = \beta' B' \in [B],\ C = \gamma' C' \in [C],\ D = \delta' D' \in [D] \tag{5.54}$$

with $\alpha', \beta', \gamma', \delta' \in \mathbb{F} \setminus \{0\}$. Inserting the expressions of equation (5.54) into equation (5.52),

$$\gamma\gamma' C' = \alpha' A' + \lambda\beta' B', \qquad \delta\delta' D' = \alpha' A' + \mu\beta' B', \tag{5.55}$$

and dividing by α',

$$\frac{\gamma\gamma'}{\alpha'} C' = A' + \lambda\frac{\beta'}{\alpha'} B', \qquad \frac{\delta\delta'}{\alpha'} D' = A' + \mu\frac{\beta'}{\alpha'} B', \tag{5.56}$$

we get

$$CR([A][B]\,[C][D]) = \frac{\lambda}{\mu} = CR(AB\,CD). \tag{5.57}$$

Theorem 5.16. *Let*

$$CR(AB\,CD) = \frac{\lambda}{\mu} =: \sigma \tag{5.58}$$

denote the cross ratio of the four different basic elements A, B, C and D according to Definition 5.15. We then have

$$CR(AB\,CD) = \sigma \qquad CR(AB\,DC) = \frac{1}{\sigma} \tag{5.59}$$

$$CR(AC\,DB) = \frac{1}{1-\sigma} \qquad CR(AC\,BD) = 1-\sigma \tag{5.60}$$

$$CR(AD\,BC) = \frac{\sigma-1}{\sigma} \qquad CR(AD\,CB) = \frac{\sigma}{\sigma-1} \tag{5.61}$$

$$CR(BC\,DA) = \frac{\sigma}{\sigma-1} \qquad CR(BC\,AD) = \frac{\sigma-1}{\sigma} \tag{5.62}$$

$$CR(BD\,AC) = 1-\sigma \qquad CR(BD\,CA) = \frac{1}{1-\sigma} \tag{5.63}$$

$$CR(BA\,CD) = \frac{1}{\sigma} \qquad CR(BA\,CD) = \sigma \tag{5.64}$$

$$CR(CD\,AB) = \sigma \qquad CR(CD\,BA) = \frac{1}{\sigma} \tag{5.65}$$

$$CR(CA\,BD) = \frac{1}{1-\sigma} \qquad CR(CA\,DB) = 1-\sigma \tag{5.66}$$

$$CR(CB\,DA) = \frac{\sigma-1}{\sigma} \qquad CR(CB\,AD) = \frac{\sigma}{\sigma-1} \tag{5.67}$$

$$CR(DA\,BC) = \frac{\sigma}{\sigma-1} \qquad CR(DA\,BC) = \frac{\sigma-1}{\sigma} \tag{5.68}$$

$$CR(DB\,CA) = 1-\sigma \qquad CR(DB\,AC) = \frac{1}{1-\sigma} \tag{5.69}$$

$$CR(DC\,AB) = \frac{1}{\sigma} \qquad CR(DC\,BA) = \sigma \tag{5.70}$$

Proof. We will first proof the cross ratios

$$CR(AB\,DC), \qquad CR(AC\,DB) \qquad \text{and} \qquad CR(BC\,DA). \tag{5.71}$$

$CR(AB\,DC)$: The switch of the two dividing elements follows directly from Definition 5.15,

$$CR(AB\,DC) = \frac{1}{CR(AB\,CD)}. \tag{5.72}$$

$CR(AC\,DB)$: We have to determine the dividing elements D and B in terms of the two base elements A and C. From equations (5.52) we get

$$\frac{\delta\lambda}{(\lambda-\mu)}D = A + \frac{\gamma\mu}{(\lambda-\mu)}C \qquad \text{and} \qquad -\lambda B = A - \gamma C. \tag{5.73}$$

The corresponding cross ratio then is

$$\begin{aligned} CR(AC\,DB) &= \frac{\gamma\mu}{(\lambda-\mu)} \cdot \left(-\frac{1}{\gamma}\right) = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\sigma} \\ &= \frac{1}{1-CR(AB\,CD)}. \end{aligned} \tag{5.74}$$

$CR(BC\,DA)$: We have to determine the dividing elements D and A in terms of the two base elements B and C. From equations (5.52) we get

$$\frac{\delta}{(\mu-\lambda)}D = B + \frac{\gamma}{(\mu-\lambda)}C \qquad \text{and} \qquad -\frac{1}{\lambda}A = B - \frac{\gamma}{\lambda}C. \tag{5.75}$$

The corresponding cross ratio then is

$$\begin{aligned} CR(BC\,DA) &= \frac{\gamma}{(\mu-\lambda)} \cdot \left(-\frac{\lambda}{\gamma}\right) = \frac{\lambda}{\lambda-\mu} = \frac{\sigma}{\sigma-1} \\ &= \frac{CR(AB\,CD)}{CR(AB\,CD)-1}. \end{aligned} \tag{5.76}$$

With respect to the initial cross ratio $CR(ABCD)$, the three permutations of equation (5.71) generate the remaining 20 permutations.

The generating permutation of equation (5.72) leads from the left to the right side in each of the equations (5.59) to (5.70).

The generating permutation of equation (5.74) leads from the left side of the equations (5.59 to the left side of the equations (5.60, from the left side of the equations (5.60 to the left side of the equations (5.61 and from the left side of the equations (5.61 back to the left side of the equations (5.59. And similar for the groups of equations (5.62) to (5.64), (5.65) to (5.67) and (5.68) to (5.70).

The generating permutation of equation (5.76) leads from the left side of the equations (5.59 to the left side of the equations (5.62, from the left side of the equations (5.62 to the left side of the equations (5.65, from the left side of the equations (5.65 to the left side of the equations (5.68 and from the left side of the equations (5.68 back to the left side of the equations (5.59. □

The 24 permutations of how the cross ratio for four fixed basic elements can be formed end up in at maximum six different numbers. In case of the harmonic cross ratio $\sigma = -1$, which will be looked at into more detail in the two examples at the end of this subsection, the six values collapse into three: -1, $\frac{1}{2}$ and 2.

Theorem 5.17. *The cross ratio of four different basic elements*

$$T_i = \lambda_i X + \mu_i Y, \qquad i \in \{1,2,3,4\}, \tag{5.77}$$

of a k-primitive geometric form is given by

$$CR\,(T_1T_2\,T_3T_4) = \frac{\left(\frac{\lambda_1\mu_3-\mu_1\lambda_3}{\lambda_2\mu_3-\mu_2\lambda_3}\right)}{\left(\frac{\lambda_1\mu_4-\mu_1\lambda_4}{\lambda_2\mu_4-\mu_2\lambda_4}\right)}. \tag{5.78}$$

Proof. We first compute T_3 and T_4 as a function of T_1 and T_2,

$$T_3 = \lambda_3 X + \mu_3 Y \overset{!}{=} \alpha T_1 + \beta T_2, \quad T_4 = \lambda_4 X + \mu_4 Y \overset{!}{=} \gamma T_1 + \delta T_2, \tag{5.79}$$

then by comparison of the coefficients get

$$\alpha = \frac{\lambda_2\mu_3 - \mu_2\lambda_3}{\lambda_2\mu_1 - \mu_2\lambda_1}, \qquad \beta = -\frac{\lambda_1\mu_3 - \mu_1\lambda_3}{\lambda_2\mu_1 - \mu_2\lambda_1}, \tag{5.80}$$

$$\gamma = \frac{\lambda_2\mu_4 - \mu_2\lambda_4}{\lambda_2\mu_1 - \mu_2\lambda_1}, \qquad \delta = -\frac{\lambda_1\mu_4 - \mu_1\lambda_4}{\lambda_2\mu_1 - \mu_2\lambda_1}, \tag{5.81}$$

and divide in the third step the two equations (5.79) by α and γ respectively,

$$\left(\frac{1}{\alpha}\right) T_3 = T_1 + \left(\frac{\beta}{\alpha}\right) T_2, \qquad \left(\frac{1}{\gamma}\right) T_4 = T_1 + \left(\frac{\delta}{\gamma}\right) T_2. \tag{5.82}$$

According to Definition 5.15 the cross ratio is

$$CR\,(T_1T_2\,T_3T_4) = \frac{\left(\frac{\beta}{\alpha}\right)}{\left(\frac{\delta}{\gamma}\right)} = \frac{\left(\frac{\lambda_1\mu_3-\mu_1\lambda_3}{\lambda_2\mu_3-\mu_2\lambda_3}\right)}{\left(\frac{\lambda_1\mu_4-\mu_1\lambda_4}{\lambda_2\mu_4-\mu_2\lambda_4}\right)}. \tag{5.83}$$

□

Theorem 5.18. *In projective geometry $\mathcal{P}_n$ intersection and connection maintain the cross ratio.*

Proof. We first confirm the assertion with respect to the operation of connection ($\wedge$) in the plus approach. By precondition the four basic elements

$$A = \langle A \rangle_k^+, \qquad B = \langle B \rangle_k^+, \qquad C = \langle C \rangle_k^+, \qquad D = \langle D \rangle_k^+, \tag{5.84}$$

of a k-primitive geometric form with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B \tag{5.85}$$

can be connected with the basic element $Z = \langle Z \rangle_l^+$, $l \in \mathbb{N}$, $0 < l < n$,

$$A_1 = A \wedge Z \neq 0, \qquad B_1 = B \wedge Z \neq 0, \tag{5.86}$$

$$C_1 = C \wedge Z \neq 0, \qquad D_1 = D \wedge Z \neq 0. \tag{5.87}$$

We then have

$$\gamma C_1 = A_1 + \lambda B_1 \qquad \delta D_1 = A_1 + \mu B_1 \tag{5.88}$$

and

$$CR\,(ABCD) = CR\,(A_1 B_1 C_1 D_1)\,. \tag{5.89}$$

The assertion of Theorem 5.18 with respect to the operation of intersection ($\vee$) follows with the major principle of duality (Theorem 5.9). □

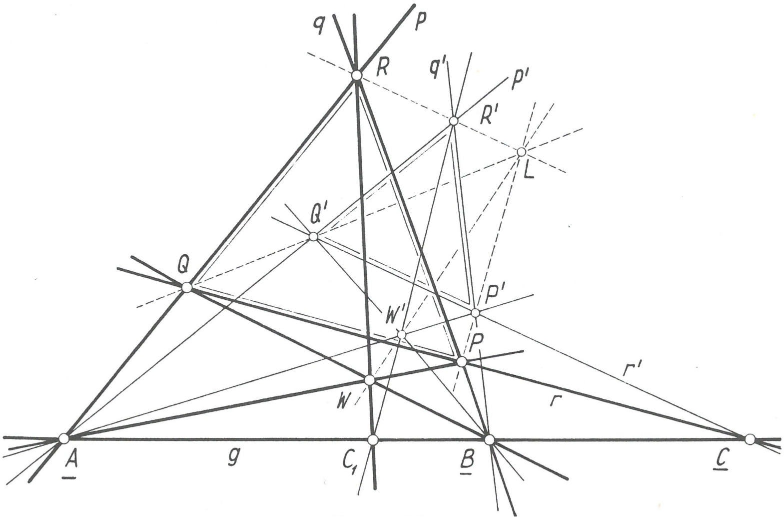

FIGURE 1. Two four-points $QRPW$ and $Q'R'P'W'$ sharing the same harmonic set of points $ABCC_1$. This drawing is a copy of Figure 155 from [LE70, p. 159].

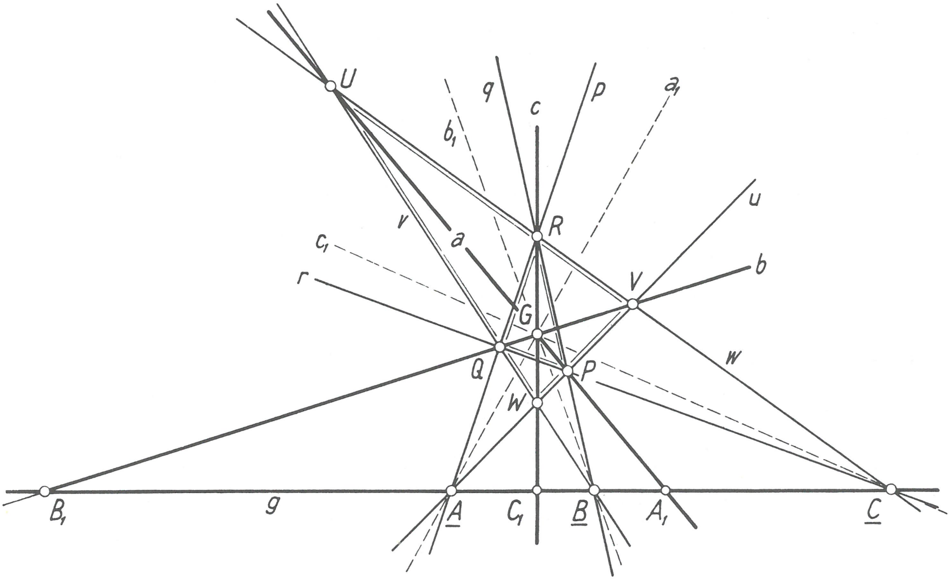

FIGURE 2. Given are the three fixed points A, B, C on the line g. The four-point $QRPW$ determines the harmonic point set $AB\,CC_1$. The four-point $QURP$ determines the harmonic point set $BC\,AA_1$. And the four-point $QRVP$ determines the harmonic point set $CA\,BB_1$. This drawing is a copy of Figure 154 from [LE70, p. 158].

Examples.

(A) Harmonic Set $AB\,CC_1$ of Four Points on a Line

We are working in the projective plane $\mathcal{P}_3$. The set of four points A, B, C, C_1 is said to be *harmonic*, if and only if there is a complete four-point[13], of which one pair of opposite sides goes through the first base point A, another pair of opposite sides goes through the second base point B, the fifth side goes through the dividing point C and the sixth side goes through the dividing point C_1. Whenever the points A, B, C are fixed and the points A and B are chosen to be basic points of the cross ratio, independent of which complete four-point we are using to construct the fourth harmonic point C_1, the latter will be the same. Figure 1 provides an example for this fact.

Let us compute the cross ratio of the harmonic point-set $AB\,CC_1$. For this we choose

$$A := P_{001}, \qquad B := P_{010}, \qquad Q := P_{100}, \tag{5.90}$$

---

[13] A four-point in the projective plane is consisting out of four points in general position plus its six connecting lines. The latter are naturally divided into three pairs of opposite lines. [LE70, p. 160], [Edw03, p. 50] See also Figure 1.

$$P := P_{001} + P_{010} + P_{100}.$$

The point C is lying on the pencil of points $X = \alpha A + \beta B$ as well as on the pencil of points $X = \gamma Q + \delta P$. We thus get

$$\begin{aligned} C &= \alpha A + \beta B = \gamma Q + \delta P \qquad \alpha, \beta, \gamma, \delta \in \mathbb{F} \setminus \{0\} && (5.91) \\ &= \alpha P_{001} + \beta P_{010} = \gamma P_{100} + \delta (P_{001} + P_{010} + P_{100}), \\ \iff \ & 0 = (\alpha - \delta) P_{001} + (\beta - \delta) P_{010} - (\gamma + \delta) P_{100}, && (5.92) \\ \iff \ & \alpha = \delta, \quad \beta = \delta, \quad \gamma = -\delta, && (5.93) \\ \iff \ & C \simeq A + B = P_{001} + P_{010}. && (5.94) \end{aligned}$$

In a similar way one can compute

$$W \simeq B + Q = P_{010} + P_{100}, \qquad R \simeq A + Q = P_{001} + P_{100}, \tag{5.95}$$

$$C_1 \simeq A - B = P_{001} - P_{010}. \tag{5.96}$$

With equations 5.90, 5.94, 5.96 and 5.53 we get the cross ratio of the harmonic point-set to be

$$\begin{aligned} CR(AB\,CC_1) &= CR(P_{100} P_{010}\,(P_{001} + P_{010})(P_{001} - P_{010})) && (5.97) \\ &= -1. \end{aligned}$$

Exchanging the dividing points C and C_1 in the cross ratio leads to

$$\begin{aligned} CR(AB\,C_1C) &= CR(P_{100} P_{010}\,(P_{001} - P_{010})(P_{001} + P_{010})) && (5.98) \\ &= \frac{1}{CR(AB\,CC_1)} = -1. \end{aligned}$$

(B) The Harmonic Point-Sets $AB\,CC_1$, $BC\,AA_1$ and $CA\,BB_1$

There are three different harmonic point-sets with respect to three fixed points A, B, C on a line g. Figure 2 provides an example. Out of A, B, C we can choose three different pairs of base points. And for each choice of the base points, the third given point is one of the dividing points. The second dividing point has to be determined.

As in Example (A), we choose our basis points and the 'unity' point according to equation (5.90). The points C, W, R and C_1 are then given by equations (5.94) to (5.96). In order to determine the remaining dividing points A_1 and B_1, we first compute the points U and V.

$$\begin{aligned} U &= \alpha B + \beta Q = \gamma C + \delta R \qquad \alpha, \beta, \gamma, \delta \in \mathbb{F} \setminus \{0\} && (5.99) \\ &= \alpha P_{010} + \beta P_{100} = \gamma (P_{001} + P_{010}) + \delta (P_{001} + P_{100}), \\ \iff \ & 0 = -(\gamma + \delta) P_{001} + (\alpha - \gamma) P_{010} + (\beta - \delta) P_{100}, && (5.100) \\ \iff \ & \alpha = \gamma, \quad \beta = \delta, \quad \gamma = -\delta, && (5.101) \\ \iff \ & U \simeq B - Q = P_{010} - P_{100}. && (5.102) \end{aligned}$$

$$\begin{aligned} V &= \alpha A + \beta W = \gamma C + \delta R \qquad \alpha, \beta, \gamma, \delta \in \mathbb{F} \setminus \{0\} && (5.103) \\ &= \alpha P_{001} + \beta (P_{010} + P_{100}) = \gamma (P_{001} + P_{010}) + \delta (P_{001} + P_{100}), \\ \iff \ & 0 = (\alpha - \gamma - \delta) P_{001} + (\beta - \gamma) P_{010} + (\beta - \delta) P_{100}, && (5.104) \end{aligned}$$

$$\begin{aligned} &\iff \alpha = \gamma + \delta, \quad \gamma = \beta, \quad \delta = \beta, && (5.105)\\ &\iff V \simeq 2A + W = 2P_{001} + P_{010} + P_{100}. && (5.106)\end{aligned}$$

With U and V we compute B_1 and A_1.

$$\begin{aligned} B_1 &= \alpha A + \beta B = \gamma Q + \delta V \qquad \alpha, \beta, \gamma, \delta \in \mathbb{F} \setminus \{0\} && (5.107)\\ &= \alpha P_{001} + \beta P_{010} = \gamma P_{100} + \delta(2P_{001} + P_{010} + P_{100}), \\ \iff 0 &= (\alpha - 2\delta)P_{001} + (\beta - \delta)P_{010} - (\gamma + \delta)P_{100}, && (5.108)\\ \iff \alpha &= 2\delta, \quad \beta = \delta, \quad \gamma = -\delta, && (5.109)\\ \iff B_1 &\simeq 2A + B = 2P_{001} + P_{010}. && (5.110)\end{aligned}$$

$$\begin{aligned} A_1 &= \alpha A + \beta B = \gamma U + \delta P \qquad \alpha, \beta, \gamma, \delta \in \mathbb{F} \setminus \{0\} && (5.111)\\ &= \alpha P_{001} + \beta P_{010} = \gamma(P_{010} - P_{100}) + \delta(P_{001} + P_{010} + P_{100}), \\ \iff 0 &= (\alpha - \delta)P_{001} + (\beta - \gamma - \delta)P_{010} + (\gamma - \delta)P_{100}, && (5.112)\\ \iff \alpha &= \delta, \quad \beta = \gamma + \delta, \quad \gamma = \delta, && (5.113)\\ \iff B_1 &\simeq A + 2B = P_{001} + 2P_{010}. && (5.114)\end{aligned}$$

The corresponding harmonic cross ratios then are (while fixing the base point pairs in one position each)

$$\begin{aligned} CR(BC\,AA_1) &= CR(BC\,(B - C)(B + C)) = -1, && (5.115)\\ CR(CA\,BB_1) &= CR(CA\,(C - A)(C + A)) = -1. && (5.116)\end{aligned}$$

Exchanging the dividing points in the cross ratio leads to

$$\begin{aligned} CR(BC\,AA_1) &= CR(BC\,(B + C)(B - C)) && (5.117)\\ &= \frac{1}{CR(BC\,AA_1)} = -1, \\ CR(CA\,BB_1) &= CR(CA\,(C + A)(C - A)) && (5.118)\\ &= \frac{1}{CR(CA\,BB_1)} = -1. \end{aligned}$$

Using other permutaions of the harmonic point-sets $AB\,CC_1$, $BC\,AA_1$ and $CA\,BB_1$ as the six displayed ones, we would get for the harmonic cross ratio not only -1, but also 2 and $\frac{1}{2}$.

6. Projective Geometry. Projective Transformations

Projective transformations are represented by algebra isomorphisms as defined in Section 4. Algebra isomorphisms transform elements of projective geometry $\mathcal{P}_n$ one-to-one from Λ_n to Λ'_n.

Theorem 6.1. *If, according to Notation 4.22,*

$$\begin{array}{llll} \phi : & \Lambda_n & \longrightarrow & \Lambda'_n \\ & N & \longmapsto & M' = \phi(N) \end{array} \qquad (6.1)$$

represents an algebra isomorphism, then the latter also maps the corresponding classes one-to-one,

$$[M'] = \phi([N])\,. \tag{6.2}$$

Proof. Any multi vector $X \in \Lambda_n$ of the class $[N]$, i. e.

$$X \in \Lambda_n, \qquad X = \sum_{k=0}^{n} \xi_k \langle N\rangle_k \in [N], \qquad \xi_k \in \mathbb{F} \setminus \{0\}, \tag{6.3}$$

is transformed by ϕ to the class $[M']$,

$$\phi(X) \in [M']. \tag{6.4}$$

And any multi vector $Y' \in \Lambda'_n$ of the class $[M']$, i. e.

$$Y' \in \Lambda'_n, \qquad Y' = \sum_{k=0}^{n} \xi'_k \langle M'\rangle_k \in [M'], \qquad \xi'_k \in \mathbb{F} \setminus \{0\}, \tag{6.5}$$

is transformed by ϕ^{-1} to the class $[N]$,

$$\phi^{-1}(Y') \in [N]. \tag{6.6}$$

□

Based on Theorem 6.1 it makes sense to define projective transformations in terms of algebra isomorphisms.

Definition 6.2 (Projective Transformation, Collineation, Correlation, Projectivity). Let ϕ be an algebra isomorphism according to Notation 4.22.

In the context of projective geometry $\mathcal{P}_n$ any algebra isomorphism ϕ represents a *projective transformation.* In case ϕ is an *even* algebra isomorphism, the corresponding projective transformation is also called *collineation*; in case ϕ is an *odd* algebra isomorphism, the corresponding projective transformation is also called *correlation.* If ϕ is an algebra automorphism, the corresponding projective transformation is also called *projectivity.*

In case ϕ is a *non-regular* algebra homomorphism, i. e. $\det \phi = 0$, it may represent a *degenerate* projective transformation, a *degenerate* collineation, a *degenerate* correlation or a *degenerate* projectivity respectively.

Definition and Theorem 6.3 (Equivalent Algebra Isomorphisms). With a, according to Notation 4.22, given algebra isomorphism

$$\begin{array}{llll} \phi: & \Lambda_n & \longrightarrow & \Lambda'_n \\ & X & \longmapsto & Y' = \phi(X) \end{array} \tag{6.7}$$

also any *equivalent* algebra isomorphism $\phi_{\simeq}$ defined by

$$\begin{array}{llll} \phi_{\simeq}: & \Lambda_n & \longrightarrow & \Lambda'_n \\ & X & \longmapsto & Y' = \phi_{\simeq}(X) \end{array} \tag{6.8}$$

$$:= \sum_{k=0}^{n} \xi'_k\, \phi(\langle X\rangle_k), \quad \xi'_k \in \mathbb{F} \setminus \{0\}, \tag{6.9}$$

represents the same projective transformation as ϕ does. $\phi_{\simeq}$ and ϕ belong to the same *class* of algebra isomorphisms,

$$\phi_{\simeq}, \phi \in [\phi]. \tag{6.10}$$

Proof. This is a consequence of how geometric elements are defined in projective geometry $\mathcal{P}_n$. Compare paragraph (A1c) of Definition 5.3. □

Theorem 6.4. *Projective transformations*

$$\begin{array}{rlcl} \phi: & \Lambda_n & \longrightarrow & \Lambda'_n \\ & X & \longmapsto & Y' = \phi(X) \end{array} \tag{6.11}$$

are one-to-one transformations, maintain incidence and the cross ratio.

Proof. Projective transformations are one-to-one transformations by Definition 6.2. They are represented by algebra isomorphisms. Equivalent classes are transformed by the algebra isomorphisms one-to-one, too.

Two elements $[A]$ and $[B]$ coincide if and only if

$$\left.\begin{array}{rcl} \langle A\rangle_k \wedge \langle B\rangle_l & = & 0, \\ \langle A\rangle_k \vee \langle B\rangle_l & = & 0, \end{array}\right\} \quad \forall\, k, l \in \{0, 1, \dots, n\}. \tag{6.12}$$

Compare paragraph (A2) of Definition 5.3. Since the exterior products are maintained by any algebra isomorphism according to Theorem 4.38, projective transformations maintain the incidence of elements.

If the four basic elements A, B, C and D of equation (5.27) form the cross ratio $CR(ABCD) = \lambda/\mu$ of equation (5.28), then also the transformed basic elements

$$\gamma\phi(C) = \phi(A) + \lambda\phi(B) \qquad \text{and} \qquad \delta\phi(D) = \phi(A) + \mu\phi(B) \tag{6.13}$$

do so,

$$CR\,(\phi(A)\phi(B)\,\phi(C)\phi(D)) = \frac{\lambda}{\mu} = CR(AB\,CD). \tag{6.14}$$

And with equation (5.32) we know, it is also true for the involved classes, i. e. the projective basic elements,

$$CR\,([\phi(A)][\phi(B)]\,[\phi(C)][\phi(D)]) = \frac{\lambda}{\mu} = CR([A][B]\,[C][D]). \tag{6.15}$$

□

Theorem 6.5 (Groups of Projectivities). *The set P of all projectivities*

$$\begin{array}{rlcl} \phi: & \Lambda_n & \longrightarrow & \Lambda_n \\ & X & \longmapsto & Y = \phi(X) \end{array} \tag{6.16}$$

forms a group $(P,\ )$ with respect to concatenation.

The set P_{col} of all collineations contained in P represents, again with respect to concatenation, a real subgroup $(P_{col},\ ) \subset (P,\ )$.

Proof. Compare Theorem 4.44. □

Theorem 6.6 (Fundamental Theorem of Projective Geometry). *A projective transformation according to Definition 6.2,*

$$\begin{array}{llll} \phi: & \Lambda_n & \longrightarrow & \Lambda'_n \\ & X & \longmapsto & Y' = \phi(X), \end{array} \tag{6.17}$$

is determined by $n+1$ pairs of basic elements from $\mathcal{P}_n$ and $\mathcal{P}'_n$ respectively,

$$(X_i, Y'_i), \qquad i \in \{1, 2, \ldots, n+1\}, \quad n \in \mathbb{N}, \tag{6.18}$$

with

$$\phi(X_i) \overset{!}{=} \sigma'_i Y'_i, \qquad \sigma'_i \in \mathbb{F} \setminus \{0\}. \tag{6.19}$$

In addition, the basic elements

$$X_i = \sum_{S(\mathbf{b})=1} \nu_{\mathbf{b}}^{(i)} B_{\mathbf{b}} \in \Lambda_n, \qquad Y'_i = \sum_{S(\mathbf{b})=1} \xi_{\mathbf{b}}^{\prime(i)} B'_{\mathbf{b}} \in \Lambda'_n \tag{6.20}$$

of the sets

$$\mathcal{X} := \{X_i \mid i \in \{1, 2, \ldots, n+1\}, \quad n \in \mathbb{N}\}, \tag{6.21}$$

$$\mathcal{Y}' := \{Y'_i \mid i \in \{1, 2, \ldots, n+1\}, \quad n \in \mathbb{N}\}, \tag{6.22}$$

each are required to be in general position according to Definition 5.13.

Proof. Since the basic elements of the sets $\mathcal{X}$ and $\mathcal{Y}'$ are in general position, we can write

$$X_{n+1} = \sum_{i=1}^{n} \epsilon_i X_i, \qquad Y'_{n+1} = \sum_{i=1}^{n} \theta'_i Y'_i, \tag{6.23}$$

where, according to Theorem 5.14, all coefficients ϵ_i or θ'_i do not vanish. On the level of 1-vectors, the projective transformation takes on the form

$$\phi(B_{\mathbf{b}}) = \sum_{S(c)=1} \kappa_{\mathbf{bc}} B'_{\mathbf{c}}, \qquad S(\mathbf{b}) = 1. \tag{6.24}$$

We want to determine the coefficients of the matrix

$$\underline{K}_{\bar{1}} := (\kappa_{\mathbf{bc}})_{S(\mathbf{b})=1; S(\mathbf{c})=1} \tag{6.25}$$

which is equal to one of the matrices $\underline{B}_{\bar{1}}$, $\hat{\underline{B}}_{\bar{1}}$ of equation (4.94) or to one of the matrices $\underline{\Gamma}_{\bar{1}}$, $\hat{\underline{\Gamma}}_{\bar{1}}$ of equation (4.95) depending on the choice of bases $\{B_{\mathbf{b}}\}$ and $\{B'_{\mathbf{b}}\}$. The first n pairs (X_i, Y'_i) deliver the n^2 conditions

$$\phi(X_i) = \sum_{\substack{S(\mathbf{b}) = 1 \\ S(\mathbf{c}) = 1}} \nu_{\mathbf{b}}^{(i)} \kappa_{\mathbf{bc}} B'_{\mathbf{c}} \overset{!}{=} \sigma'_i \sum_{S(\mathbf{c})=1} \xi_{\mathbf{c}}^{\prime(i)} B'_{\mathbf{c}} \tag{6.26}$$

or

$$\sum_{S(\mathbf{b})=1} \nu_{\mathbf{b}}^{(i)} \kappa_{\mathbf{bc}} \overset{!}{=} \sigma'_i \xi_{\mathbf{c}}^{\prime(i)}, \qquad i \in \{1, 2, \ldots, n\}, \quad S(\mathbf{c}) = 1, \tag{6.27}$$

for the n^2 coefficients of the matrix $\underline{K}_{\overline{1}}$. The remaining n coefficients σ'_i from the first n equations (6.15) are determined with the last pair (X_{n+1}, Y'_{n+1}),

$$\phi(X_{n+1}) = \sum_{i=1}^{n} \epsilon_i \phi(X_i) = \sum_{i=1}^{n} \epsilon_i \sigma'_i Y'_i \overset{!}{=} \sigma'_{n+1} Y'_{n+1} = \sigma_{n+1} \sum_{i=1}^{n} \theta'_i Y'_i, \quad (6.28)$$

with the result

$$\sigma'_i = \epsilon_i^{-1} \theta'_i \sigma'_{n+1}. \quad (6.29)$$

□

Examples.

(E1) Construction of a point-wise conic section in the planar field $\mathcal{P}_3$ through a projective transformation from a pencil of lines A to a pencil of lines B

Let the points A, B, C and D of the planar field $\mathcal{P}_3$ in the plus approach be in general position, i. e.

$$A \wedge B \wedge C \neq \mathbf{0}, \qquad B \wedge C \wedge D \neq \mathbf{0}, \quad (6.30)$$

$$C \wedge D \wedge A \neq \mathbf{0}, \qquad D \wedge A \wedge B \neq \mathbf{0}. \quad (6.31)$$

The three connecting lines through the points A or B respectively are denoted by

$$a := -A \wedge C \qquad b := -B \wedge C \qquad c := A \wedge B \quad (6.32)$$

$$d_A := A \wedge D \qquad d_B := B \wedge D \quad (6.33)$$

The line d_A belongs to the pencil of lines A, the line d_B belongs to the pencil of lines B. We can thus choose

$$d_A = a + c \qquad d_B = b + c. \quad (6.34)$$

According to the Fundamental Theorem 6.6, a projective transformation from the pencil of lines A to the pencil of lines B

$$\begin{array}{llcl} \Pi : & \Lambda_2 \subset \Lambda_3^+ & \longrightarrow & \Lambda'_2 \subset \Lambda_3^+ \\ & A : \; x = \lambda a + \mu c & \longmapsto & B : \; y = \Pi(x) \\ & & & \quad = \lambda \Pi(a) + \mu \Pi(c) \end{array} \quad (6.35)$$

is given by three pairs, e. g.

$$(a, \Pi(a)) \quad \text{with } \Pi(a) = \sigma_1 \cdot c, \quad (6.36)$$

$$(c, \Pi(c)) \quad \text{with } \Pi(c) = \sigma_2 \cdot b, \quad (6.37)$$

$$(d_A, \Pi(d_A)) \quad \text{with } \Pi(d_A) = \sigma_3 \cdot d_B. \quad (6.38)$$

From equations (6.36) to (6.38) we get

$$\Pi(d_A) = \Pi(a) + \Pi(c) = \sigma_1\, c + \sigma_2\, b \overset{!}{=} \sigma_3\, d_B = \sigma_3(b + c) \quad (6.39)$$

and, thus, with

$$\sigma_1 = \sigma_2 = \sigma_3 \quad (6.40)$$

the projective transformation Π

$$\begin{aligned} \Pi: \quad & \Lambda_2 \subset \Lambda_3^+ && \longrightarrow && \Lambda_2' \subset \Lambda_3^+ \\ & A: \; x = \lambda a + \mu c && \longmapsto && B: \; y = \Pi(x) = \sigma_3(\lambda c + \mu b). \end{aligned} \tag{6.41}$$

Since the pencils of lines A and B both belong to the planar field Λ_3^+, corresponding lines x and $\Pi(x)$ meet in the point

$$X = \alpha A + \beta B + \gamma C \tag{6.42}$$

with

$$0 = x \wedge X \qquad \text{and} \qquad 0 = \Pi(x) \wedge X. \tag{6.43}$$

In case of $\lambda, \mu \neq 0$ the conditions of the equations (6.43),

$$\begin{aligned} \mathbf{0} &= (\lambda a + \mu c) \wedge (\alpha A + \beta B + \gamma C) = \lambda\beta \cdot a \wedge B + \mu\gamma \cdot c \wedge C \\ &= (-\lambda\beta + \mu\gamma) A \wedge B \wedge C, \end{aligned} \tag{6.44}$$

$$\begin{aligned} \mathbf{0} &= (\mu b + \lambda c) \wedge (\alpha A + \beta B + \gamma C) = \mu\alpha \cdot b \wedge A + \lambda\gamma \cdot c \wedge C \\ &= (-\mu\alpha + \lambda\gamma) A \wedge B \wedge C, \end{aligned} \tag{6.45}$$

lead to

$$\alpha = \frac{\lambda}{\mu}\gamma \qquad \text{and} \qquad \beta = \frac{\mu}{\lambda}\gamma \tag{6.46}$$

and, provided that $\gamma \neq 0$, to the parametrisation of second order for the point-wise conic section

$$\begin{aligned} X &= \alpha A + \beta B + \gamma C = \frac{\lambda}{\mu}\gamma A + \frac{\mu}{\lambda}\gamma B + \gamma C \\ &\simeq \lambda^2 A + \mu^2 B + \lambda\mu C. \end{aligned} \tag{6.47}$$

In case of $\gamma = 0$, the points $X = \alpha A + \beta B \neq \mathbf{0}$ would belong to the line c. The line c intersects the constructed conic section in the points A and B. The parametrisation of the point-wise conic section in equation (6.47) yields point A in case the parameters take on the values $\lambda \neq 0$ and $\mu = 0$ and point B in case the parameters take on the values $\lambda = 0$ and $\mu \neq 0$. This is why the parametrisation of the point-wise conic section in equation (6.47) holds for all values of λ and μ.

Since point D is laying by precondition on the lines $d_A = a + c$ and the lines $d_B = b + c$, we have

$$D = A + B + C \qquad \text{with} \qquad \lambda = \mu = 1. \tag{6.48}$$

with

Each position of line x in the pencil of lines A determines one and only one point X. The set of all points X forms a point-wise conic section in the planar field Λ_3^+, which is touching line a in point A, which is touching line b in point B, and to which the point D is belonging.

We summarise the result in

Theorem 6.7 (Point-wise Conics in the Planar Field $\mathcal{P}_3$). *If the four points*

$$A, \qquad B, \qquad C \qquad \text{and} \qquad D \tag{6.49}$$

of the planar field $\mathcal{P}_3$ are in general position and if the three sides of the triangle ABC are denoted as follows

$$a = A \wedge C, \qquad b = -B \wedge C, \qquad c = A \wedge B, \tag{6.50}$$

then the point-wise conic section, which is touching line a in point A, which is touching line b in point B, and to which the point D is belonging with the parameters $\lambda = \mu = 1$, is described by the parametrisation of second order

$$X(\lambda, \mu) = \lambda^2 A + \mu^2 B + \lambda\mu C. \tag{6.51}$$

Proof. See Example (E1) above. □

(E2) The tangents of a point-wise conic section in the planar field $\mathcal{P}_3$

Theorem 6.8 (Tangents to a Parametrised Point-wise Curve in $\mathcal{P}_n$). *Let*

$$\begin{array}{llll} X: & \mathbb{F} & \longrightarrow & \Lambda_n^{1+} \\ & t & \longmapsto & X(t) \end{array} \tag{6.52}$$

represent a point-wise curve in $\mathcal{P}_n$ depending on the parameter t. If the curve $X(t)$ is differentiable with respect to t, then

$$\begin{array}{llll} x: & \mathbb{F} & \longrightarrow & \Lambda_n^{2+} \\ & t & \longmapsto & x(t) := X(t) \wedge \dfrac{dX(t)}{dt} \end{array} \tag{6.53}$$

describes the tangent $x(t)$ in the point $X(t)$ to the point-wise curve of equation (6.52).

Proof.

$$\begin{aligned} x(t) &= \lim_{\Delta t \to 0} [X(t) \wedge X(t + \Delta t)] \\ &\simeq \lim_{\Delta t \to 0} \left[\frac{X(t) \wedge (X(t + \Delta t) - X(t))}{\Delta t} \right] \\ &= X(t) \wedge \lim_{\Delta t \to 0} \left[\frac{(X(t + \Delta t) - X(t))}{\Delta t} \right] \\ &= X(t) \wedge \frac{dX(t)}{dt} \end{aligned} \tag{6.54}$$

□

In order to find the tangents to the point-wise conic section of Theorem 6.7, we apply Theorem 6.8 to the parametrisation of the conic section in equation 6.51. Fix the parameter μ to a non-zero value and take λ as the varying parameter.

$$x(\lambda, \mu) = X(\lambda, \mu) \wedge \frac{dX(\lambda, \mu)}{d\lambda} \tag{6.55}$$

$$
\begin{aligned}
&= (\lambda^2 A + \mu^2 B + \lambda\mu\, C) \wedge (2\lambda\, A + \mu\, C)\\
&= -(2\lambda\mu^2 c + 2\lambda^2 \mu\, a - \lambda^2 \mu\, a + \mu^3 b)\\
&\simeq \lambda^2 a + \mu^2 b + 2\lambda\mu\, c
\end{aligned}
$$

Equation (6.55) is also valid in the case $\mu = 0$ and $\lambda \neq 0$, since it delivers the tangent a for the point A.

(E3) Construction of a line-wise conic section in the planar field $\mathcal{P}_3$ through a projective transformation from a pencil of points a to a pencil of points b

Theorem 6.9 (Line-wise Conics in the Planar Field $\mathcal{P}_3$). *If the four lines*

$$a, \qquad b, \qquad c \qquad \text{and} \qquad d \tag{6.56}$$

of the planar field $\mathcal{P}_3$ are in general position and if the three corners of the triangle abc are denoted as follows

$$A = a \vee c, \qquad B = -b \vee c, \qquad C = a \vee b, \tag{6.57}$$

then the line-wise conic section, which touches the point of tangency A with the line a, which touches the point of tangency B with the line b, and to which the line d is belonging with the parameters $\lambda = \mu = 1$, is described by the parametrisation of second order

$$x(\lambda, \mu) = \lambda^2 a + \mu^2\, b + \lambda\mu\, c. \tag{6.58}$$

Proof. Apply the major principle of duality of Theorem 5.9 to Theorem 6.7. □

The underlying projective transformation Φ from the pencil of points a to the pencil of points b is

$$
\begin{array}{llcl}
\Phi: & \Lambda_2 \subset \Lambda_3^- & \longrightarrow & \Lambda_2' \subset \Lambda_3^- \\
& a:\ X = \lambda A + \mu C & \longmapsto & b:\ Y = \Phi(X) = \sigma_3(\lambda C + \mu B).
\end{array} \tag{6.59}
$$

(E4) The points of tangency of a line-wise conic section in the planar field $\mathcal{P}_3$

Theorem 6.10 (Points of Tangency to a Parametrised Plane-wise Curve in $\mathcal{P}_n$). *Let*

$$
\begin{array}{llcl}
X: & \mathbb{F} & \longrightarrow & \Lambda_n^{1-} \\
& t & \longmapsto & X(t)
\end{array} \tag{6.60}
$$

represent a plane-wise curve in $\mathcal{P}_n$ depending on the parameter t. If the curve $X(t)$ is differentiable with respect to t, then

$$
\begin{array}{llcl}
x: & \mathbb{F} & \longrightarrow & \Lambda_n^{2-} \\
& t & \longmapsto & x(t) := X(t) \vee \dfrac{dX(t)}{dt}
\end{array} \tag{6.61}
$$

describes the instantaneous axis of rotation $x(t)$ of the plane $X(t)$ along the plane-wise curve of equation (6.60).

Proof. Apply the major principle of duality of Theorem 5.9 to Theorem 6.8. □

In order to find the points of tangency to the line-wise conic section of Theorem 6.9, we apply Theorem 6.10 adjusted to the field $\mathcal{P}_3$. In this case $X(t) \in \Lambda_3^{1-}$ represents a line-wise curve in the field $\mathcal{P}_3$, i. e. the parametrisation of the conic section in equation 6.58. And

$$x(t) = X(t) \vee \frac{dX(t)}{dt} \tag{6.62}$$

describes the curve of instantaneous points of tangency. Again, we fix the parameter μ to a non-zero value and take λ as the varying parameter.

$$\begin{aligned} x(\lambda,\mu) &= X(\lambda,\mu) \vee \frac{dX(\lambda,\mu)}{d\lambda} \\ &= (\lambda^2 a + \mu^2 b + \lambda\mu\, c) \vee (2\lambda\, a + \mu\, c) \\ &= -(2\lambda\mu^2 C + 2\lambda^2\mu\, A - \lambda^2\mu\, A + \mu^3 B) \\ &\simeq \lambda^2 A + \mu^2 B + 2\lambda\mu\, C \end{aligned} \tag{6.63}$$

Equation (6.63) is also valid in the case $\mu = 0$ and $\lambda \neq 0$, since it delivers the point of tangency A for the line a.

7. Projective Geometry. Quadrics and Orthonormal Bases

In Section 8 we will do the transition from exterior double algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ to Clifford double algebra $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$. Clifford double algebra Γ_n will, in addition to the properties of projective algebra, carry the imprint of a Clifford algebra twice.

As a preparation, we review and provide in this section the concepts of polarities, quadratic and bilinear forms, quadrics, and orthonormal bases in the context of projective geometry $\mathcal{P}_n$. In addition, we introduce, what a pair of naturally associated — non-degenerate and degenerate — polarities is. We define harmonic orthonormal systems of bases and list the quadrics in $\mathcal{P}_2$, $\mathcal{P}_3$ as well as in $\mathcal{P}_4$.

7.1. Polarities

Definition 7.1 (Polarities). In projective geometry $\mathcal{P}_n$, the odd algebra endomorphisms

$$\begin{array}{cccc} \hat{\pi}: & \Lambda_n & \longrightarrow & \Lambda_n \\ & X & \longmapsto & Y = \hat{\pi}(X) \end{array} \tag{7.1}$$

or

$$\begin{array}{cccc} \hat{\rho}: & \Lambda_n & \longrightarrow & \Lambda_n \\ & X & \longmapsto & Y = \hat{\rho}(X) \end{array} \tag{7.2}$$

represent, eventually degenerate, correlations. Compare Definition 6.2. These correlations become *polarities*, if the non-vanishing parts of their corresponding transformations, applied twice, are projective identity mappings.

In the non-degenerate case, the transformations (7.1) and (7.2) then have to fulfill the conditions

$$\hat{\pi}^2(X) = \varepsilon_k X \qquad \forall\ X \in \Lambda_n^k \text{ and } k \in \{0, 1, \ldots, n\}, \tag{7.3}$$

$$\hat{\rho}^2(X) = \varepsilon'_k X \qquad \forall\ X \in \Lambda_n^k \text{ and } k \in \{0, 1, \ldots, n\} \tag{7.4}$$

with

$$\varepsilon_k, \varepsilon'_k \in \mathbb{F} \setminus \{0\} \tag{7.5}$$

and in the cases of $k = 0$ and $k = n$ we get

$$\varepsilon_0 = \frac{1}{\lambda^2}, \quad \varepsilon'_0 = \frac{1}{\mu^2}, \quad \varepsilon_n = (\lambda^{n-1} \det \hat{\pi})^2, \quad \varepsilon'_n = (\mu^{n-1} \det \hat{\rho})^2. \tag{7.6}$$

The defining conditions for the degenerate polarities are

$$\hat{\pi}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{0+} := \Lambda_n^{0+}, \tag{7.7}$$

$$\hat{\rho}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{0-} := \Lambda_n^{0-}, \tag{7.8}$$

$$\hat{\pi}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{k+} \subset \Lambda_n^{k+}, \quad k \in \{1, \ldots, n-1\}, \tag{7.9}$$

$$\hat{\pi}^2(X) = \varepsilon_k X \qquad \forall\ X \in U_n^{k+} \subset \Lambda_n^{k+}, \quad \varepsilon_k \in \mathbb{F} \setminus \{0\}, \tag{7.10}$$

$$\hat{\rho}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{k-} \subset \Lambda_n^{k-}, \quad k \in \{1, \ldots, n-1\}, \tag{7.11}$$

$$\hat{\rho}^2(X) = \varepsilon'_k X \qquad \forall\ X \in U_n^{k-} \subset \Lambda_n^{k-}, \quad \varepsilon'_k \in \mathbb{F} \setminus \{0\}, \tag{7.12}$$

$$\hat{\pi}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{n+} := \Lambda_n^{n+}, \tag{7.13}$$

$$\hat{\rho}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{n-} := \Lambda_n^{n-} \tag{7.14}$$

with the sub vector spaces N_n^k and U_n^k satisfying

$$N_n^k \oplus U_n^k = \Lambda_n^k, \tag{7.15}$$

$$\dim N_n^k + \dim U_n^k = \dim \Lambda_n^k, \qquad 1 \leq \dim N_n^k \leq \dim \Lambda_n^k = \binom{n}{k}. \tag{7.16}$$

The conditions for the degenerate polarities of equations (7.7) to (7.14) read in a shorter and more compact version as follows,

$$\hat{\pi}^2(X) = \varepsilon_k X \qquad \forall\ X \in U_n^{k+} \subset \Lambda_n^{k+}, \quad k \in \{1, \ldots, n-1\}, \tag{7.17}$$

$$\hat{\pi}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{k+} \subset \Lambda_n^{k+}, \quad k \in \{0, 1, \ldots, n-1, n\}, \tag{7.18}$$

$$\hat{\rho}^2(X) = \varepsilon'_k X \qquad \forall\ X \in U_n^{k-} \subset \Lambda_n^{k-}, \quad k \in \{1, \ldots, n-1\}, \tag{7.19}$$

$$\hat{\rho}^2(X) = \mathbf{0} \qquad \forall\ X \in N_n^{k-} \subset \Lambda_n^{k-}, \quad k \in \{0, 1, \ldots, n-1, n\}. \tag{7.20}$$

The matrices $\underline{\hat{B}}_{\overline{k}}$ of the algebra homomorphism $\hat{\pi}$ and the matrices $\underline{\hat{\Gamma}}_{\overline{k}}$ of the algebra homomorphism $\hat{\rho}$ — compare Notation 4.16 — need to satisfy certain conditions, if they should represent polarities. As we will see in the following Theorem 7.2, these matrices are either symmetric or skew-symmetric in the non-degenerate cases. The same is true for the non-degenerate parts of the degenerate polarities. In detail we have

Theorem 7.2 (Matrix of a Polarity). *The algebra endomorphisms* $\hat{\pi}$ *or* $\hat{\rho}$ *represent non-degenerate polarities in the sense of Definition 7.1, if and only if the corresponding matrices* $\underline{\hat{B}}_{\overline{k}}$ *or* $\underline{\hat{\Gamma}}_{\overline{k}}$ *satisfy the conditions*

$$(-1)^{k(n-k)}\underline{\hat{B}}_{\overline{k}}\operatorname{cof}(\underline{\hat{B}}_{\overline{k}}) = \varepsilon_k\,\mathbb{I}_{\binom{n}{k}}, \tag{7.21}$$

$$(-1)^{k(n-k)}\underline{\hat{\Gamma}}_{\overline{k}}\operatorname{cof}(\underline{\hat{\Gamma}}_{\overline{k}}) = \varepsilon'_k\,\mathbb{I}_{\binom{n}{k}} \tag{7.22}$$

$$\underline{\hat{B}}_{\overline{k}}^{T} = (-1)^{k(n-k)}\frac{\varepsilon_k}{\lambda^{n-2}\det\hat{\pi}}\,\underline{\hat{B}}_{\overline{k}}, \tag{7.23}$$

$$\underline{\hat{\Gamma}}_{\overline{k}}^{T} = (-1)^{k(n-k)}\frac{\varepsilon'_k}{\mu^{n-2}\det\hat{\rho}}\,\underline{\hat{\Gamma}}_{\overline{k}}, \tag{7.24}$$

$$\underline{\hat{B}}_{\overline{k}}^{-1} = (-1)^{k(n-k)}\frac{1}{\varepsilon_k}\operatorname{cof}(\underline{\hat{B}}_{\overline{k}}), \tag{7.25}$$

$$\underline{\hat{\Gamma}}_{\overline{k}}^{-1} = (-1)^{k(n-k)}\frac{1}{\varepsilon'_k}\operatorname{cof}(\underline{\hat{\Gamma}}_{\overline{k}}). \tag{7.26}$$

respectively for all $k \in \{0, 1, \dots, n\}$*. In addition we have*

$$\varepsilon_k = \pm\,\lambda^{n-2}\det\hat{\pi}, \qquad\qquad \varepsilon'_k = \pm\,\mu^{n-2}\det\hat{\rho}, \tag{7.27}$$

which means, the matrices $\underline{\hat{B}}_{\overline{k}}$ *and* $\underline{\hat{\Gamma}}_{\overline{k}}$ *are either symmetric or skew-symmetric.*

In case of a degenerate polarity $\hat{\pi}$ *with* $\det\hat{\pi} = 0$ *or* $\hat{\rho}$ *with* $\det\hat{\rho} = 0$ *we get for all* $k \in \{0, 1, \dots, n\}$ *respectively,*

$$(-1)^{k(n-k)}\underline{\hat{B}}_{\overline{k}}\operatorname{cof}(\underline{\hat{B}}_{\overline{k}})X = \mathbf{0} \quad \forall\; X \in N_n^{k+}, \tag{7.28}$$

$$(-1)^{k(n-k)}\underline{\hat{B}}_{\overline{k}}\operatorname{cof}(\underline{\hat{B}}_{\overline{k}})X = \varepsilon_k X \quad \forall\; X \in U_n^{k+}, \tag{7.29}$$

$$(-1)^{k(n-k)}\underline{\hat{\Gamma}}_{\overline{k}}\operatorname{cof}(\underline{\hat{\Gamma}}_{\overline{k}}) = \mathbf{0} \quad \forall\; X \in N_n^{k-}, \tag{7.30}$$

$$(-1)^{k(n-k)}\underline{\hat{\Gamma}}_{\overline{k}}\operatorname{cof}(\underline{\hat{\Gamma}}_{\overline{k}}) = \varepsilon'_k X \quad \forall\; X \in U_n^{k-}. \tag{7.31}$$

The non-vanishing part of the matrix $\underline{\hat{B}}_{\overline{k}}$ *or the matrix* $\underline{\hat{\Gamma}}_{\overline{k}}$ *is either symmetric or skew-symmetric and with it the whole matrix, too.*

Proof. Let $X = \sum_{S(\mathbf{b})=k}\mu_{\mathbf{b}}P_{\mathbf{b}}$ be a homogeneous multi vector of grade k in the plus approach. We then have in case of a polarity from the plus to the minus approach $\hat{\pi}$,

$$\hat{\pi}^2(X) = \sum_{S(\mathbf{b})=k}\mu_{\mathbf{b}}\,\hat{\pi}^2(P_{\mathbf{b}}) = \sum_{\substack{S(\mathbf{b})=k\\ S(\mathbf{c})=k}}\mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bc}}\,\hat{\pi}(E_{\mathbf{c}}) \tag{7.32}$$

$$= \sum_{\substack{S(\mathbf{b})=k\\ S(\mathbf{c})=k}}\alpha_{\mathbf{c}\overline{\mathbf{c}}}\mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bc}}\,\hat{\pi}(P_{\overline{\mathbf{c}}}) = \sum_{\substack{S(\mathbf{b})=k\\ S(\mathbf{c})=k\\ S(\mathbf{d})=k}}\alpha_{\mathbf{c}\overline{\mathbf{c}}}\mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bc}}\hat{\beta}_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\,E_{\overline{\mathbf{d}}}$$

$$
\begin{aligned}
&= \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \alpha_{\mathbf{c}\overline{\mathbf{c}}}\alpha_{\overline{\mathbf{d}}\mathbf{d}}\mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bc}}\hat{\beta}_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\, P_{\mathbf{d}} \\
&= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bc}} \left(\alpha_{\overline{\mathbf{c}}\mathbf{c}}\alpha_{\overline{\mathbf{d}}\mathbf{d}}\hat{\beta}_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\right) P_{\mathbf{d}} \\
&= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bc}} \left(\operatorname{cof}(\underline{\hat{B}}_{\overline{k}})\right)_{\mathbf{cd}} P_{\mathbf{d}} \\
&\overset{!}{=} \begin{cases} \mathbf{0}, & \text{in case } \det\hat{\pi} = 0 \text{ and } X \in N_n^{k+}, \\ \varepsilon_k X, & \text{otherwise,} \end{cases}
\end{aligned}
$$

if and only if equations (7.21), (7.28) and (7.29) are true.

Let $X = \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}} E_{\mathbf{b}}$ be a homogeneous multi vector of grade k in the minus approach. We then have in case of a polarity from the minus to the plus approach $\hat{\rho}$,

$$
\begin{aligned}
\hat{\rho}^2(X) &= \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}}\, \hat{\rho}^2(E_{\mathbf{b}}) = \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k}} \nu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bc}}\, \hat{\rho}(P_{\mathbf{c}}) \\
&= \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k}} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\nu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bc}}\, \hat{\rho}(E_{\overline{\mathbf{c}}}) = \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\nu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bc}}\hat{\gamma}_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\, P_{\overline{\mathbf{d}}} \\
&= \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \alpha_{\overline{\mathbf{c}}\mathbf{c}}\alpha_{\mathbf{d}\overline{\mathbf{d}}}\nu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bc}}\hat{\gamma}_{\overline{\mathbf{c}}\overline{\mathbf{d}}}\, E_{\mathbf{d}} \\
&= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \nu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bc}}(\alpha_{\overline{\mathbf{c}}\mathbf{c}}\alpha_{\overline{\mathbf{d}}\mathbf{d}}\hat{\gamma}_{\overline{\mathbf{c}}\overline{\mathbf{d}}})E_{\mathbf{d}} \\
&= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k \\ S(\mathbf{d}) = k}} \nu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bc}} \left(\operatorname{cof}(\underline{\hat{\Gamma}}_{\overline{k}})\right)_{\mathbf{cd}} E_{\mathbf{d}} \\
&\overset{!}{=} \begin{cases} \mathbf{0}, & \text{in case } \det\hat{\rho} = 0 \text{ and } X \in N_n^{k-}, \\ \varepsilon'_k X, & \text{otherwise,} \end{cases}
\end{aligned}
\tag{7.33}
$$

if and only if equations (7.22), (7.30) and (7.31) are true.

The equations (7.23) to (7.26) follow from equations (7.21) and (7.22) respectively. The equations (7.27) are a consequence of equations (7.23) and (7.24) respectively.

On the bases of equations (7.23), (7.24) and (7.27) we may conclude the non-degenerate matrices $\underline{\hat{B}}_{\bar{k}}$ and $\underline{\hat{\Gamma}}_{\bar{k}}$ to be either symmetric or skew-symmetric.

On the bases of equations (7.29) and (7.31), the equations (7.23) and (7.24) are true too in case of degenerate matrices $\underline{\hat{B}}_{\bar{k}}$ and $\underline{\hat{\Gamma}}_{\bar{k}}$ for all $X \in U_n^k$. The conditions of equations (7.28) and (7.30) determine the remaining parts of the matrices $\underline{\hat{B}}_{\bar{k}}$ and $\underline{\hat{\Gamma}}_{\bar{k}}$ to be null sub matrices. Thus, altogether, the degenerate matrices $\underline{\hat{B}}_{\bar{k}}$ and $\underline{\hat{\Gamma}}_{\bar{k}}$ are either symmetric or skew-symmetric, too. □

Definition 7.3 (Null Polarities, Non-Null Polarities). Polarities $\hat{\pi}_{\text{skew}}$ and $\hat{\rho}_{\text{skew}}$ with a skew-symmetric matrix with respect to the 1-vectors,

$$\hat{\pi}^T_{\text{skew}}(X_{\bar{1}}) = -\hat{\pi}_{\text{skew}}(X_{\bar{1}}), \quad \hat{\rho}^T_{\text{skew}}(X_{\bar{1}}) = -\hat{\rho}_{\text{skew}}(X_{\bar{1}}) \quad \forall\, X_{\bar{1}} \in \Lambda^1_n, \tag{7.34}$$

are called *null polarities.* Polarities $\hat{\pi}_{\text{sym}}$ and $\hat{\rho}_{\text{sym}}$ with a symmetric matrix with respect to the 1-vectors,

$$\hat{\pi}^T_{\text{sym}}(X_{\bar{1}}) = \hat{\pi}_{\text{sym}}(X_{\bar{1}}), \qquad \hat{\rho}^T_{\text{sym}}(X_{\bar{1}}) = \hat{\rho}_{\text{sym}}(X_{\bar{1}}) \qquad \forall\, X_{\bar{1}} \in \Lambda^1_n, \tag{7.35}$$

are called *non-null polarities* or just *polarities.*

Corollary 7.4. *A null polarity $\hat{\pi}_{\text{skew}}$ or $\hat{\rho}_{\text{skew}}$ maps 1-elements of the plus or minus approach to 1-elements of the minus or plus approach respectively. In case of null polarities from the plus to the minus approach $\hat{\pi}_{\text{skew}}$, the 1-element $X^+_{\bar{1}}$ coincides with its corresponding 1-element $\hat{\pi}_{\text{skew}}(X^+_{\bar{1}})^-_{\bar{1}}$ for all $X^+_{\bar{1}} \in \Lambda^{1+}_n$,*

$$X^+_{\bar{1}} \wedge \hat{\pi}_{\text{skew}}(X^+_{\bar{1}})^-_{\bar{1}} = \mathbf{0}, \qquad \forall\, X^+_{\bar{1}} \in \Lambda^{1+}_n, \tag{7.36}$$

$$X^+_{\bar{1}} \vee \hat{\pi}_{\text{skew}}(X^+_{\bar{1}})^-_{\bar{1}} = \mathbf{0}. \tag{7.37}$$

The same ist true for polarities from the minus to the plus approach $\hat{\rho}_{\text{skew}}$,

$$\hat{\rho}_{\text{skew}}(X^-_{\bar{1}})^+_{\bar{1}} \vee X^-_{\bar{1}} = \mathbf{0}, \qquad \forall\, X^-_{\bar{1}} \in \Lambda^{1-}_n, \tag{7.38}$$

$$\hat{\rho}_{\text{skew}}(X^-_{\bar{1}})^+_{\bar{1}} \wedge X^-_{\bar{1}} = \mathbf{0}. \tag{7.39}$$

Proof.

$$\begin{aligned} X^+_{\bar{1}} \wedge \hat{\pi}_{\text{skew}}(X^+_{\bar{1}})^-_{\bar{1}} &= \left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}} \right) \wedge \left(\sum_{\substack{S(\mathbf{c}) = 1 \\ S(\mathbf{d}) = 1}} \lambda_{\mathbf{c}} \hat{\beta}_{\mathbf{cd}} E_{\mathbf{d}} \right) \\ &= \left(\sum_{\substack{S(\mathbf{b}) = 1 \\ S(\mathbf{c}) = 1}} \lambda_{\mathbf{b}} \lambda_{\mathbf{c}} \hat{\beta}_{\mathbf{cb}} \right) \mathbf{I}^+ \end{aligned} \tag{7.40}$$

$$
\begin{aligned}
&= \left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}^2 \hat{\beta}_{\mathbf{bb}} + \sum_{\substack{S(\mathbf{b})=1 \\ S(\mathbf{c})=1 \\ \mathbf{b}<\mathbf{c}}} \lambda_{\mathbf{b}} \lambda_{\mathbf{c}} (\hat{\beta}_{\mathbf{cb}} + \hat{\beta}_{\mathbf{bc}}) \right) \mathbf{I}^+ \\
&= \mathbf{0},
\end{aligned}
$$

$$
\begin{aligned}
X_{\bar{1}}^+ \vee \hat{\pi}_{\text{skew}}(X_{\bar{1}}^+)_{\bar{1}}^- &= \left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}} \right) \vee \left(\sum_{\substack{S(\mathbf{c})=1 \\ S(\mathbf{d})=1}} \lambda_{\mathbf{c}} \hat{\beta}_{\mathbf{cd}} E_{\mathbf{d}} \right) \\
&= \left(\sum_{\substack{S(\mathbf{b})=1 \\ S(\mathbf{c})=1}} \lambda_{\mathbf{b}} \lambda_{\mathbf{c}} \hat{\beta}_{\mathbf{cb}} \right) \mathbf{I}^- \\
&= \left(\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}^2 \hat{\beta}_{\mathbf{bb}} + \sum_{\substack{S(\mathbf{b})=1 \\ S(\mathbf{c})=1 \\ \mathbf{b}<\mathbf{c}}} \lambda_{\mathbf{b}} \lambda_{\mathbf{c}} (\hat{\beta}_{\mathbf{cb}} + \hat{\beta}_{\mathbf{bc}}) \right) \mathbf{I}^- \\
&= \mathbf{0},
\end{aligned}
\tag{7.41}
$$

$$
\begin{aligned}
\hat{\rho}_{\text{skew}}(X_{\bar{1}}^-)_{\bar{1}}^+ \vee X_{\bar{1}}^- &= \left(\sum_{\substack{S(\mathbf{c})=1 \\ S(\mathbf{d})=1}} \mu_{\mathbf{c}} \hat{\gamma}_{\mathbf{cd}} P_{\mathbf{d}} \right) \vee \left(\sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}} E_{\mathbf{b}} \right) \\
&= \left(\sum_{\substack{S(\mathbf{b})=1 \\ S(\mathbf{c})=1}} \mu_{\mathbf{b}} \mu_{\mathbf{c}} \hat{\gamma}_{\mathbf{cb}} \right) \mathbf{I}^- \\
&= \left(\sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}}^2 \hat{\gamma}_{\mathbf{bb}} + \sum_{\substack{S(\mathbf{b})=1 \\ S(\mathbf{c})=1 \\ \mathbf{b}<\mathbf{c}}} \mu_{\mathbf{b}} \mu_{\mathbf{c}} (\hat{\gamma}_{\mathbf{cb}} + \hat{\gamma}_{\mathbf{bc}}) \right) \mathbf{I}^- \\
&= \mathbf{0},
\end{aligned}
\tag{7.42}
$$

$$
\hat{\rho}_{\text{skew}}(X_{\bar{1}}^-)_{\bar{1}}^+ \wedge X_{\bar{1}}^- = \left(\sum_{\substack{S(\mathbf{c})=1 \\ S(\mathbf{d})=1}} \mu_{\mathbf{c}} \hat{\gamma}_{\mathbf{cd}} P_{\mathbf{d}} \right) \wedge \left(\sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}} E_{\mathbf{b}} \right)
\tag{7.43}
$$

$$= \left(\sum_{\substack{S(\mathbf{b}) = 1 \\ S(\mathbf{c}) = 1}} \mu_{\mathbf{b}} \mu_{\mathbf{c}} \hat{\gamma}_{\mathbf{cb}} \right) \mathbf{I}^{+}$$

$$= \left(\sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}}^2 \hat{\gamma}_{\mathbf{bb}} + \sum_{\substack{S(\mathbf{b}) = 1 \\ S(\mathbf{c}) = 1 \\ \mathbf{b} < \mathbf{c}}} \mu_{\mathbf{b}} \mu_{\mathbf{c}} (\hat{\gamma}_{\mathbf{cb}} + \hat{\gamma}_{\mathbf{bc}}) \right) \mathbf{I}^{+}$$

$$= \mathbf{0},$$

□

For the remainder of this article we do not use the notation any more introduced for null and non-null polarities in Definition 7.3. We skip the indices *skew* and *sym*, since out of the context it will be clear, whether the odd algebra endomorphisms $\hat{\pi}$ and $\hat{\rho}$ refer to null, to non-null polarities or to both.

7.2. Quadratic and Bilinear Forms with Respect to Polarities

We are going to define quadratic and bilinear forms in this subsection with respect to a generic pair of non-null polarities $p = (\hat{\pi}, \hat{\rho})$.

Definition and Theorem 7.5 (Quadratic and Bilinear Forms with Respect to a Pair of Polarities). Let $\hat{\pi}$ be a generic non-null polarity from the plus to the minus approach and let $\hat{\rho}$ be a generic non-null polarity from the minus to the plus approach. Then with respect to the pair $(\hat{\pi}, \hat{\rho})$ we define the quadratic and bilinear forms as follows: In the plus approach, the *quadratic forms* by

$$\begin{aligned} Q_{\hat{\pi}}^{k+} : \quad \Lambda_n^{k+} & \longrightarrow \mathbb{F} \\ X_{\overline{k}} = \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}} & \longmapsto Q_{\hat{\pi}}^{k+}(X_{\overline{k}}) \end{aligned} \tag{7.44}$$

for $k \in \{0, 1, \ldots, n\}$ with

$$Q_{\hat{\pi}}^{k+}(X_{\overline{k}}) \mathbf{Z}^{+} := \begin{cases} \mu_{\mathbf{b}} \hat{\beta}_{\mathbf{bb}} \mu_{\mathbf{b}} \mathbf{Z}^{+}, & S(\mathbf{b}) = k = 0 \\ X_{\overline{k}} \vee \hat{\pi}(X_{\overline{k}}), & 0 < S(\mathbf{b}) = k < n \\ \mu_{\mathbf{b}} \hat{\beta}_{\mathbf{bb}} \mu_{\mathbf{b}} \mathbf{Z}^{+}, & S(\mathbf{b}) = k = n \end{cases} \tag{7.45}$$

$$= \left(\sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k}} \mu_{\mathbf{b}} \hat{\beta}_{\mathbf{bc}} \mu_{\mathbf{c}} \right) \mathbf{Z}^{+};$$

in the minus approach, the *quadratic forms* by

$$Q_{\hat{\rho}}^{k-} : \quad \Lambda_n^{k-} \longrightarrow \mathbb{F} \tag{7.46}$$

$$X_{\overline{k}} = \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}} E_{\mathbf{b}} \qquad \longmapsto \qquad Q^{k-}_{\hat{\rho}}(X_{\overline{k}})$$

for $k \in \{0, 1, \ldots, n\}$ with

$$Q^{k-}_{\hat{\rho}}(X_{\overline{k}})\mathbf{Z}^- := \begin{cases} \nu_{\mathbf{b}} \hat{\gamma}_{\mathbf{bb}} \nu_{\mathbf{b}} \mathbf{Z}^-, & S(\mathbf{b}) = k = 0 \\ X_{\overline{k}} \wedge \hat{\rho}(X_{\overline{k}}), & 0 < S(\mathbf{b}) = k < n \\ \nu_{\mathbf{b}} \hat{\gamma}_{\mathbf{bb}} \nu_{\mathbf{b}} \mathbf{Z}^-, & S(\mathbf{b}) = k = n \end{cases} \tag{7.47}$$

$$= \left((-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k}} \nu_{\mathbf{b}} \hat{\gamma}_{\mathbf{bc}} \nu_{\mathbf{c}} \right) \mathbf{Z}^-;$$

in the plus approach, the *symmetric bilinear forms* are defined by

$$\begin{aligned} B^{k+}_{\hat{\pi}} : \quad & \Lambda^{k+}_n \times \Lambda^{k+}_n & \longrightarrow \quad & \mathbb{F} \\ & (X_{\overline{k}}, Y_{\overline{k}}) & \longmapsto \quad & B^{k+}_{\hat{\pi}}(X_{\overline{k}}, Y_{\overline{k}}) \end{aligned} \tag{7.48}$$

for $X_{\overline{k}} = \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}}$, $Y_{\overline{k}} = \sum_{S(\mathbf{c})=k} \nu_{\mathbf{c}} P_{\mathbf{c}}$ and $k \in \{0, 1, \ldots, n\}$ with

$$B^{k+}_{\hat{\pi}}(X_{\overline{k}}, Y_{\overline{k}})\mathbf{Z}^+ := \begin{cases} \mu_{\mathbf{b}} \hat{\beta}_{\mathbf{bb}} \nu_{\mathbf{b}} \mathbf{Z}^+, & S(\mathbf{b}) = k = 0 \\ \frac{1}{2}(X_{\overline{k}} \vee \hat{\pi}(Y_{\overline{k}}) + Y_{\overline{k}} \vee \hat{\pi}(X_{\overline{k}})), & 0 < S(\mathbf{b}) = k < n \\ \mu_{\mathbf{b}} \hat{\beta}_{\mathbf{bb}} \nu_{\mathbf{b}} \mathbf{Z}^+, & S(\mathbf{b}) = k = n \end{cases}$$

$$= \left(\frac{1}{2} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k}} \mu_{\mathbf{b}} (\hat{\beta}_{\mathbf{bc}} + \hat{\beta}_{\mathbf{cb}}) \nu_{\mathbf{c}} \right) \mathbf{Z}^+ \tag{7.49}$$

and in the minus approach by

$$\begin{aligned} B^{k-}_{\hat{\rho}} : \quad & \Lambda^{k-}_n \times \Lambda^{k-}_n & \longrightarrow \quad & \mathbb{F} \\ & (X_{\overline{k}}, Y_{\overline{k}}) & \longmapsto \quad & B^{k-}_{\hat{\rho}}(X_{\overline{k}}, Y_{\overline{k}}) \end{aligned} \tag{7.50}$$

for $X_{\overline{k}} = \sum_{S(\mathbf{b})=k} \zeta_{\mathbf{b}} E_{\mathbf{b}}$, $Y_{\overline{k}} = \sum_{S(\mathbf{c})=k} \xi_{\mathbf{c}} E_{\mathbf{c}}$ and $k \in \{0, 1, \ldots, n\}$ with

$$B^{k-}_{\hat{\rho}}(X_{\overline{k}}, Y_{\overline{k}})\mathbf{Z}^- := \begin{cases} \zeta_{\mathbf{b}} \hat{\gamma}_{\mathbf{bb}} \xi_{\mathbf{b}} \mathbf{Z}^-, & S(\mathbf{b}) = k = 0 \\ \frac{1}{2}(X_{\overline{k}} \wedge \hat{\rho}(Y_{\overline{k}}) + Y_{\overline{k}} \wedge \hat{\rho}(X_{\overline{k}})), & 0 < S(\mathbf{b}) = k < n \\ \zeta_{\mathbf{b}} \hat{\gamma}_{\mathbf{bb}} \xi_{\mathbf{b}} \mathbf{Z}^-, & S(\mathbf{b}) = k = n \end{cases}$$

$$= \left(\frac{1}{2} (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b}) = k \\ S(\mathbf{c}) = k}} \zeta_{\mathbf{b}} (\hat{\gamma}_{\mathbf{bc}} + \hat{\gamma}_{\mathbf{cb}}) \xi_{\mathbf{c}} \right) \mathbf{Z}^-. \tag{7.51}$$

The quadratic and bilinear forms are related by

$$B^k_p(X_{\overline{k}}, Y_{\overline{k}}) = \frac{1}{2} \left(Q^k_p(X_{\overline{k}} + Y_{\overline{k}}) - Q^k_p(X_{\overline{k}}) - Q^k_p(Y_{\overline{k}}) \right), \tag{7.52}$$

$$Q^k_p(X_{\overline{k}}) = B^k_p(X_{\overline{k}}, X_{\overline{k}}), \tag{7.53}$$

where we are using the letter p as index for the pair $p = (\hat{\pi}, \hat{\rho})$, since the equations (7.52) and (7.53) hold in the plus and in the minus approach. In the plus approach we have $p = \hat{\pi}$, in the minus approach $p = \hat{\rho}$.

Proof. For $S(\mathbf{b}) = S(\mathbf{c}) = k$ with $0 < k < n$ we have

$$P_{\mathbf{b}} \vee \hat{\pi}(P_{\mathbf{c}}) = \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}} \vee \left(\sum_{S(\mathbf{d})=k} \hat{\beta}_{\mathbf{cd}} E_{\mathbf{d}} \right) = \alpha^2_{\overline{\mathbf{b}}\mathbf{b}} \hat{\beta}_{\mathbf{cb}} \mathbf{I}^- = \hat{\beta}_{\mathbf{cb}} \mathbf{Z}^+, \tag{7.54}$$

$$\begin{aligned} E_{\mathbf{b}} \wedge \hat{\rho}(E_{\mathbf{c}}) &= \alpha_{\mathbf{b}\overline{\mathbf{b}}} P_{\overline{\mathbf{b}}} \wedge \left(\sum_{S(\mathbf{d})=k} \hat{\gamma}_{\mathbf{cd}} P_{\mathbf{d}} \right) = \alpha_{\overline{\mathbf{b}}\mathbf{b}} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \hat{\gamma}_{\mathbf{cb}} \mathbf{I}^+ \\ &= (-1)^{k(n-k)} \hat{\gamma}_{\mathbf{cb}} \mathbf{Z}^-. \end{aligned} \tag{7.55}$$

Using the expressions of equations (7.54) and (7.55) in the defining equations (7.45), (7.47), (7.49) and (7.51), we confirm the latter to be correct.

The forms Q^k_p are quadratic, i. e. $Q^k_p(\lambda X) = \lambda^2 Q^k_p(X)$, and the bilinear forms B^k_p are symmetric and bilinear by definition. Equations (7.52) and (7.53) follow directly. □

Corollary 7.6. *Let a generic non-null polarity from the plus to the minus approach $\hat{\pi}$ and a generic non-null polarity from the minus to the plus approach $\hat{\rho}$ form a pair of polarities $p = (\hat{\pi}, \hat{\rho})$. Then its* quadratic forms Q^k_p *and its* bilinear forms B^k_p *evaluate on the bases vectors as follows,*

$$Q^{k+}_{\hat{\pi}}(P_{\mathbf{b}}) = \hat{\beta}_{\mathbf{bb}}, \qquad S(\mathbf{b}) = k, \tag{7.56}$$

$$Q^{k-}_{\hat{\rho}}(E_{\mathbf{b}}) = (-1)^{k(n-k)} \hat{\gamma}_{\mathbf{bb}}, \tag{7.57}$$

$$B^{k+}_{\hat{\pi}}(P_{\mathbf{b}}, P_{\mathbf{c}}) = \frac{1}{2} \left(\hat{\beta}_{\mathbf{bc}} + \hat{\beta}_{\mathbf{cb}} \right), \qquad S(\mathbf{c}) = k, \tag{7.58}$$

$$B^{k-}_{\hat{\rho}}(E_{\mathbf{b}}, E_{\mathbf{c}}) = \frac{1}{2} (-1)^{k(n-k)} \left(\hat{\gamma}_{\mathbf{bc}} + \hat{\gamma}_{\mathbf{cb}} \right). \tag{7.59}$$

Proof. Insert the bases vectors into equations (7.45), (7.47), (7.49) and (7.51) of Definition and Theorem 7.5. □

7.3. Quadrics

As it is well known, quadrics may be defined in terms of quadratic forms. We will repreat this definition here for the two approaches to projective algebra $\Lambda^{\pm}_n$.

Definition 7.7 (Quadrics of Non-Null Polarities). Let $\hat{\pi}$ be a generic non-null polarity from the plus to the minus approach and let $\hat{\rho}$ be a generic non-null polarity from the minus to the plus approach. Its quadratic forms $Q^{k\pm}_p$ are given by Definition and Theorem 7.5. Then the set of k-vectors

$$\mathcal{Q}^{k\pm}_p := \left\{ X \in \Lambda^{k\pm}_n \mid Q^{k\pm}_p(X) = 0 \right\} \tag{7.60}$$

is called $(k\pm)$-*quadric* of the polarity $\hat{\pi}$ or $\hat{\rho}$ respectively, or simply $(k\pm)$-*quadric.* In case the polarity $\hat{\pi}$ or $\hat{\rho}$ is degenerate, the respective $(k\pm)$-quadric is called *degenerate*; otherwise *non-degenerate.*

If from the context it is clear, which approach is addressed, the plus minus notation can be omitted. We then speak, for example, of a *non-degenerate k-quadric* or a pair of two *k-quadrics.*

The different grades k of the k-quadrics $\mathcal{Q}_p^{k\pm}$ may be summarized in one single expression $\mathcal{Q}_p^{\pm}$.

Please note, for a generic pair of polarities $(\hat{\pi}, \hat{\rho})$, in general, we get *two* different quadrics $\mathcal{Q}_{\hat{\pi}}^{+}$ and $\mathcal{Q}_{\hat{\rho}}^{-}$.

7.4. Orthonormal Bases

We rephrase here the definition of an orthogonal or orthonormal basis of a vector space V with respect to a given, eventually degenerate, quadratic form Q as well as the theorems constructing them. See e. g. Definition 2.3 and Theorems 2.2 to 2.5 in [LS16, pp. 8].

Definition 7.8 (Orthogonal and Orthonormal Basis). Let V be any vector space over the field $\mathbb{F}$ (with $\operatorname{char}\mathbb{F} \neq 2$) of finite dimension $\dim V = n$ equipped with a basis $\{B_1, B_2, \ldots, B_n\}$ and an, eventually degenerate, quadratic form Q as well as a bilinear form B. The quadratic form Q and the bilinear form B are related to each other according to the equations (7.52) and (7.53).

The vectors B_k, $k \in \{1, 2, \ldots, n\}$ form an *orthonormal* basis of the vector space V, if and only if the two following conditions hold:

(a) $B(B_j, B_k) = 0, \quad \forall\, j \neq k$ with $j, k \in \{1, 2, \ldots, n\}$,

(b) $Q(B_k) \in \{0, 1, -1\} \subset \mathbb{F}, \quad \forall\, B_k$ with $k \in \{1, 2, \ldots, n\}$.

If the basis vectors B_k satisfy only condition (a), the basis of V is called *orthogonal.*

Theorem 7.9 (Existence of an Orthogonal Basis). *Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \neq 2$ and let V be a vector space over the field $\mathbb{F}$ of finite dimension $\dim V = n$ equipped with an, eventually degenerate, quadratic form Q as well as a bilinear form B.*

There exists an orthogonal basis $\{B_1, B_2, \ldots, B_n\}$ in V with respect to the bilinear form B.

Proof. Compare Theorem 2.2 and its proof in [LS16, p. 8]. □

Definition 7.10 (Spin Field $\mathbb{F}$). A field $\mathbb{F}$ is called *spin field*, if and only if for any $\eta \in \mathbb{F} \setminus \{0\}$ there is a $\lambda \in \mathbb{F}$ with

$$\lambda^2\, \eta \overset{!}{=} \pm 1. \tag{7.61}$$

The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are spin fields.

Theorem 7.11 (Existence of an Orthonormal Basis). *Let $\mathbb{F}$ be a field with $\operatorname{char}\mathbb{F} \neq 2$ and let V be a vector space over the field $\mathbb{F}$ of finite dimension $\dim V = n$ equipped with an, eventually degenerate, quadratic form Q as well as a bilinear form B.*

If $\mathbb{F}$ is a spin field, there exists an orthonormal basis $\{B_1, B_2, \ldots, B_n\}$ in V with respect to the, eventually degenerate, quadratic form Q. In case of

$\mathbb{F} = \mathbb{C}$, the condition (b) of Definition 7.8 can be restricted to $Q(B_k) \in \{0, 1\}$ for all $k \in \{1, 2, \ldots, n\}$.

Proof. Compare Theorem 2.4, Theorem 2.5 and the corresponding proofs in [LS16, pp. 9]. □

Definition and Theorem 7.12 (Silvester's Law of Interia, Signature). Let $\mathbb{F}$ be a spin field with $\operatorname{char}\mathbb{F} \neq 2$ and let V be a vector space over the field $\mathbb{F}$ of finite dimension $\dim V = n$ equipped with an, eventually degenerate, quadratic form Q as well as a bilinear form B.

Every orthonormal basis $\{B_1, B_2, \ldots, B_n\}$ in V with respect to a, eventually degenerate, quadratic form Q contains the same number s of basis members with $Q(B_k) = 1$, the same number t of basis members with $Q(B_k) = -1$ and the same number u of basis members with $Q(B_k) = 0$ while $s + t + u = n$.

In case of $\mathbb{F} = \mathbb{C}$, according to Theorem 7.11, the number t can be put to be always zero, $t = 0$.

The triple

$$\mathcal{S} := (s, t, u) \quad \begin{cases} & \text{number of basis members } B_k \text{ with} \\ s & \to \quad Q(B_k) = 1 \\ t & \to \quad Q(B_k) = -1 \\ u & \to \quad Q(B_k) = 0 \end{cases} \tag{7.62}$$

is called the *signature* of the pair vector space V equipped with the quadratic form Q.

Proof. Compare Theorem 2.4, Theorem 2.5 and the corresponding proofs in [LS16, pp. 9]. □

Given two generic non-null polarities $\hat{\pi}$ and $\hat{\rho}$, there are now *two* different quadratic forms $Q^{1+}_{\hat{\pi}}$ and $Q^{1-}_{\hat{\rho}}$ for one and the same vector space $\Lambda_n(+, \cdot)$, i. e. there are also two different signatures $\mathcal{S}^+ = (s^+, t^+, u^+)$ and $\mathcal{S}^- = (s^-, t^-, u^-)$. As a consequence, we may represent the generic non-null polarity $\hat{\pi}$ with respect to an orthonormal basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$ and the generic non-null polarity $\hat{\rho}$ with respect to an orthonormal basis $\{E'_{\mathbf{b}}\} \subset \Lambda_n$, where with respect to the system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$, with respect to the system of bases $\{P'_{\mathbf{b}}\}, \{E'_{\mathbf{b}}\} \subset \Lambda_n$, but not necessarily with respect to the system of bases $\{P_{\mathbf{b}}\}, \{E'_{\mathbf{b}}\} \subset \Lambda_n$ we have an harmonic model of Λ_n according to Definition 4.26.

Corollary 7.13. *Let* $\mathbf{b}$ *be a binary number with* $0 \leq S(\mathbf{b}) = k \leq n$ *and let a generic non-null polarity from the plus to the minus approach*

$$\begin{array}{rlcl} \hat{\pi}: & \Lambda_n & \longrightarrow & \Lambda_n \\ & P_{\mathbf{b}} & \longmapsto & \hat{\pi}(P_{\mathbf{b}}) = \displaystyle\sum_{S(\mathbf{c})=k} \hat{\beta}_{\mathbf{bc}} E_{\mathbf{c}} \end{array} \tag{7.63}$$

and a generic non-null polarity from the minus to the plus approach

$$\begin{array}{rlcl} \hat{\rho}: & \Lambda_n & \longrightarrow & \Lambda_n \end{array} \tag{7.64}$$

$$E'_{\mathbf{b}} \qquad \longmapsto \qquad \hat{\rho}(E'_{\mathbf{b}}) = \sum_{S(\mathbf{c})=k} \hat{\gamma}_{\mathbf{bc}} P'_{\mathbf{c}}$$

form a pair of polarities $(\hat{\pi}, \hat{\rho})$*. Then with Definition and Theorem 7.12, there is an orthonormal basis* $\{P_{\mathbf{b}}\} \subset \Lambda_n$*,*

$$Q^{1+}_{\hat{\pi}}(P_{{}_l\mathbf{u}}) = \eta^{+}_{{}_l\mathbf{u}}, \qquad S(\mathbf{u}) = n, \qquad 1 \leq l \leq n, \tag{7.65}$$

with signature $\mathcal{S}^{+} = (s^{+}, t^{+}, u^{+})$*, i. e.*

$$\begin{aligned} \eta^{+}_{{}_l\mathbf{u}} &= 1, & 1 &\leq l \leq s^{+}, \\ \eta^{+}_{{}_l\mathbf{u}} &= -1, & s^{+}+1 &\leq l \leq s^{+}+t^{+}, \\ \eta^{+}_{{}_l\mathbf{u}} &= 0, & s^{+}+t^{+}+1 &\leq l \leq s^{+}+t^{+}+u^{+} = n, \end{aligned} \tag{7.66}$$

and an orthonormal basis $\{E'_{\mathbf{b}}\} \subset \Lambda_n$*,*

$$Q^{1-}_{\hat{\rho}}(E'_{{}_l\mathbf{u}}) = \eta^{-}_{{}_l\mathbf{u}}, \qquad S(\mathbf{u}) = n, \qquad 1 \leq l \leq n, \tag{7.67}$$

with signature $\mathcal{S}^{-} = (s^{-}, t^{-}, u^{-})$*, i. e.*

$$\begin{aligned} \eta^{-}_{{}_l\mathbf{u}} &= 1, & 1 &\leq l \leq s^{-}, \\ \eta^{-}_{{}_l\mathbf{u}} &= -1, & s^{-}+1 &\leq l \leq s^{-}+t^{-}, \\ \eta^{-}_{{}_l\mathbf{u}} &= 0, & s^{-}+t^{-}+1 &\leq l \leq s^{-}+t^{-}+u^{-} = n. \end{aligned} \tag{7.68}$$

With respect to the orthonormal basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$*, the matrix* $\underline{\hat{B}}$ *of* $\hat{\pi}$*, cf. equation (4.98), takes on the form*

$$\hat{\beta}_{\mathbf{bb}} = \frac{1 - \delta_{0\,\det\hat{\pi}}}{\lambda}, \qquad \text{in case } S(\mathbf{b}) = 0, \tag{7.69}$$

$$\hat{\beta}_{\mathbf{bc}} = \lambda^{k-1} \left(\prod_{l=1}^{k} \eta^{+}_{{}_l\mathbf{b}} \right) \delta_{\mathbf{bc}}, \qquad \text{in case } 1 \leq S(\mathbf{b}) = k \leq n, \tag{7.70}$$

and with respect to the orthonormal basis $\{E'_{\mathbf{b}}\} \subset \Lambda_n$*, the matrix* $\underline{\hat{\Gamma}}$ *of* $\hat{\rho}$*, cf. equation (4.99), takes on the form*

$$\hat{\gamma}_{\mathbf{bb}} = \frac{1 - \delta_{0\,\det\hat{\rho}}}{\mu}, \qquad \text{in case } S(\mathbf{b}) = 0, \tag{7.71}$$

$$\hat{\gamma}_{\mathbf{bc}} = (-1)^{k(n-1)} \mu^{k-1} \left(\prod_{l=1}^{k} \eta^{-}_{{}_l\mathbf{b}} \right) \delta_{\mathbf{bc}} \quad \text{in case } 1 \leq S(\mathbf{b}) = k \leq n. \tag{7.72}$$

Proof. For the cases $S(\mathbf{b}) = 0$ in equations (7.67) and (7.69) see the two expressions of equation (4.232) of Definition and Theorem 4.33. For $S(\mathbf{b}) = 1$ we get with Corollary 7.6

$$Q^{1+}_{\hat{\pi}}(P_{\mathbf{b}}) = \hat{\beta}_{\mathbf{bb}} = \eta^{+}_{\mathbf{b}}, \tag{7.73}$$

$$Q^{1-}_{\hat{\rho}}(E'_{\mathbf{b}}) = (-1)^{(n-1)} \hat{\gamma}_{\mathbf{bb}} = (-1)^{(n-1)} (-1)^{(n-1)} \eta^{-}_{\mathbf{b}} = \eta^{-}_{\mathbf{b}}, \tag{7.74}$$

$$B^{1+}_{\hat{\pi}}(P_{\mathbf{b}}, P_{\mathbf{c}}) = \hat{\beta}_{\mathbf{bc}} = \hat{\beta}_{\mathbf{cb}} = 0, \qquad \mathbf{b} \neq \mathbf{c}, \tag{7.75}$$

$$B^{1-}_{\hat{\rho}}(E'_{\mathbf{b}}, E'_{\mathbf{c}}) = (-1)^{(n-1)} \hat{\gamma}_{\mathbf{bc}} = (-1)^{(n-1)} \hat{\gamma}_{\mathbf{cb}} = 0, \qquad \mathbf{b} \neq \mathbf{c}. \tag{7.76}$$

For $1 \le S(\mathbf{b}) = k \le n$ we get

$$\hat{\beta}_{\mathbf{bc}} = \lambda^{k-1} \sum_{\sigma} \operatorname{sign} \sigma \prod_{l=1}^{k} \hat{\beta}_{{}_l\mathbf{b}_{\sigma(l)}\mathbf{c}} = \lambda^{k-1} \left(\prod_{l=1}^{k} \eta^{+}_{{}_l\mathbf{b}} \right) \delta_{\mathbf{bc}} \tag{7.77}$$

$$\hat{\gamma}_{\mathbf{bc}} = \mu^{k-1} \sum_{\sigma} \operatorname{sign} \sigma \prod_{l=1}^{k} \hat{\gamma}_{{}_l\mathbf{b}_{\sigma(l)}\mathbf{c}} = (-1)^{k(n-1)} \mu^{k-1} \left(\prod_{l=1}^{k} \eta^{-}_{{}_l\mathbf{b}} \right) \delta_{\mathbf{bc}} \tag{7.78}$$

Compare equations (4.83) and (4.87) of Definition and Theorem 4.15. ☐

Since, occasionally, we are using two signatures for on vector space, we need to extend the notation for the signature of a vector space.

Notation 7.14 (Vector Space V with One Signature $\mathcal{S}$ or with Two Signatures $\mathcal{S}^+$ and $\mathcal{S}^-$). In order to indicate the signature of a vector space V equipped with *one* quadratic form Q, the latter is denoted by $V_{\mathcal{S}}$ or by $V_{(s,t,u)}$ or shorter by $V_{s,t,u}$.

In case the vector space V is equipped with *two* quadratic forms Q^+ and Q^-, the latter is denoted by $V_{\mathcal{S}^+;\mathcal{S}^-}$ or by $V_{(s^+,t^+,u^+)^+;(s^-,t^-,u^-)^-}$ or shorter by $V_{s^+,t^+,u^+;s^-,t^-,u^-}$.

7.5. Pairs of Naturally Associated Polarities

So far the two non-null polarities of the pair $(\hat{\pi}, \hat{\rho})$ were arbitrary. Let us now look at pairs of non-null polarities $(\hat{\pi}, \hat{\rho})$, where both polarities are in the first step non-degenerate and, in addition, represent one and the same transformation according to Theorem 4.39.We call these pairs of non-degenerate non-null polarities $(\hat{\pi}, \hat{\rho})$ *naturally associated.*

What kind of pairs $(\hat{\pi}, \hat{\rho})$ can we choose to be naturally associated, in case the polarities $\hat{\pi}$ and $\hat{\rho}$ both are degenerate? Using the principle of continuity and following Felix Klein's ideas about the transitions between the different types of quadrics [Kle28, Kapitel II, §6, pp. 80-93], we look at limits of naturally associated pairs of non-degenerate polarities, which degenerate in the limit. This shall be our preliminary approach to *degenerate naturally associated* pairs of polarities $(\hat{\pi}, \hat{\rho})$.

Definition 7.15 (Pairs of Naturally Associated Polarities. Prelliminary Version). Two non-vanishing but otherwise generic non-null polarities $\hat{\pi}$ and $\hat{\rho}$ according to Definition 7.1 form *a pair of naturally associated polarities* $(\hat{\pi}, \hat{\rho})$, if and only if

(D1) in case one of the two polarities is non-degenerate, then both polarities $\hat{\pi}$ and $\hat{\rho}$ are non-degenerate and represent the same transformation according to Theorem 4.39; or

(D2) in case one of the two polarities is degenerate, then $\hat{\pi}$ and $\hat{\rho}$ both are degenerate and there exists a limit in the following way:

  (E1) If the degenerate polarity $\hat{\pi}$ is given (including a fixed λ, i. e. $\lambda = 1$ or $\lambda = -1$), represent $\hat{\pi}$ with respect to an orthonormal basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$ with signature

$$\mathcal{S}^+ = (s^+, t^+, u^+), \qquad 1 \le u^+ < n, \tag{7.79}$$

i. e.

$$Q^{1+}_{\hat{\pi}}(P_{{}_{l}\mathbf{u}}) = \hat{\beta}_{{}_{l}\mathbf{u}\,{}_{l}\mathbf{u}} = \eta^{+}_{{}_{l}\mathbf{u}}, \qquad S(\mathbf{u}) = n, \quad 1 \le l \le n, \tag{7.80}$$

with

$$\begin{aligned} \eta^{+}_{{}_{l}\mathbf{u}} &= 1, & & 1 \le l \le s^{+}, \\ \eta^{+}_{{}_{l}\mathbf{u}} &= -1, & & s^{+} + 1 \le l \le s^{+} + t^{+}, \\ \eta^{+}_{{}_{l}\mathbf{u}} &= 0, & & s^{+} + t^{+} + 1 \le l \le s^{+} + t^{+} + u^{+} = n. \end{aligned} \tag{7.81}$$

Else represent the degenerate polarity $\hat{\rho}$ (including a fixed μ, i. e. $\mu = 1$ or $\mu = -1$), with respect to an orthonormal basis $\{E_{\mathbf{b}}\} \subset \Lambda_n$ with signature

$$\mathcal{S}^{-} = (s^{-}, t^{-}, u^{-}), \qquad 1 \le u^{-} < n, \tag{7.82}$$

i. e.

$$Q^{1-}_{\hat{\rho}}(E_{{}_{l}\mathbf{u}}) = \eta^{+}_{{}_{l}\mathbf{u}}, \qquad S(\mathbf{u}) = n, \quad 1 \le l \le n, \tag{7.83}$$

with

$$\begin{aligned} \eta^{-}_{{}_{l}\mathbf{u}} &= 1, & & 1 \le l \le s^{-}, \\ \eta^{-}_{{}_{l}\mathbf{u}} &= -1, & & s^{-} + 1 \le l \le s^{-} + t^{-}, \\ \eta^{-}_{{}_{l}\mathbf{u}} &= 0, & & s^{-} + t^{-} + 1 \le l \le s^{-} + t^{-} + u^{-} = n. \end{aligned} \tag{7.84}$$

In the first case complement the basis $\{P_{\mathbf{b}}\}$ by a basis $\{E_{\mathbf{b}}\}$, in the second case complement the basis $\{E_{\mathbf{b}}\}$ by a basis $\{P_{\mathbf{b}}\}$, such that the system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$ is forming an harmonic model of Λ_n.

(E2) In case the degenerate polarity $\hat{\pi}$ is given, choose a polarity $\hat{\pi}_t$ depending on the parameter $t \in \mathbb{R}$ such, that with respect to $\hat{\pi}_t$ $\{P_{\mathbf{b}}\}$ becomes an orthogonal basis with the quadratic form

$$\begin{aligned} Q^{1+}_{\hat{\pi}_t}(P_{{}_{l}\mathbf{u}}) &= \hat{\beta}^{(t)}_{{}_{l}\mathbf{u}\,{}_{l}\mathbf{u}} = \varepsilon^{(t)+}_{{}_{l}\mathbf{u}} \\ &:= \begin{cases} \eta^{+}_{{}_{l}\mathbf{u}} = 1, & 1 \le l \le s^{+}, \\ \eta^{+}_{{}_{l}\mathbf{u}} = -1, & s^{+} + 1 \le l \le s^{+} + t^{+}, \\ -t, & s^{+} + t^{+} + 1 \le l \le s^{+} + t^{+} + m_1^{+}, \\ t, & n - m_2^{+} + 1 \le l \le n, \end{cases} \end{aligned} \tag{7.85}$$

$$u^{+} = m_1^{+} + m_2^{+}, \qquad \begin{cases} 0 \le m_1^{+}, m_2^{+} \le u^{+}, \\ m_1^{+}, m_2^{+} \in \mathbb{N}_0. \end{cases} \tag{7.86}$$

By construction, the polarity $\hat{\pi}_t$ is non-degenerate for $t \neq 0$ and we have

$$\lim_{t \to 0} \hat{\pi}_t = \hat{\pi}. \tag{7.87}$$

Now complement the polarity $\hat{\pi}_t$ by its naturally associated polarity $\hat{\rho}_t$. The pair $(\hat{\pi}_t, \hat{\rho}_t)$ then consists of two non-degenerate naturally associated polarities for all $t \neq 0$. (With $\det \hat{\pi}_t$ and

$\lambda \in \{1, -1\}$ also $\mu \in \{1, -1\}$ is determined, compare equations (4.281), left expression.)

The naturally associated polarity to $\hat{\pi}$ is defined by

$$\begin{aligned} \hat{\rho}(E_{\mathbf{b}}) &:= \mathbf{0}, && S(\mathbf{b}) = 0, \\ \hat{\rho}(E_{\mathbf{b}}) &:= \lim_{t \to 0} \left(\frac{\hat{\rho}_t(E_{\mathbf{b}})}{t^{(u^+ - 1)}} \right), && S(\mathbf{b}) = 1, \\ \hat{\rho}(A \vee B) &:= \mu \cdot \hat{\rho}(A) \wedge \hat{\rho}(B) && \forall\, A, B \in \Lambda_n. \end{aligned} \tag{7.88}$$

(E3) In case the degenerate polarity $\hat{\rho}$ is given, choose a polarity $\hat{\rho}_t$ depending on the parameter $t \in \mathbb{R}$ such, that with respect to $\hat{\rho}_t$ $\{E_{\mathbf{b}}\}$ becomes an orthogonal basis with the quadratic form

$$Q^{1-}_{\hat{\rho}_t}(E_{{}_l\mathbf{u}}) = (-1)^{n-1} \hat{\gamma}^{(t)}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \varepsilon^{(t)-}_{{}_l\mathbf{u}} \tag{7.89}$$

$$:= \begin{cases} \eta^-_{{}_l\mathbf{u}} = 1, & 1 \le l \le s^-, \\ \eta^-_{{}_l\mathbf{u}} = -1, & s^- + 1 \le l \le s^- + t^-, \\ -t, & s^- + t^- + 1 \le l \le s^- + t^- + m_1^-, \\ t, & n - m_2^- + 1 \le l \le n, \end{cases}$$

$$u^- = m_1^- + m_2^-, \qquad \begin{cases} 0 \le m_1^-, m_2^- \le u^-, \\ m_1^-, m_2^- \in \mathbb{N}_0. \end{cases} \tag{7.90}$$

By construction, the polarity $\hat{\rho}_t$ is non-degenerate for $t \neq 0$ and we have

$$\lim_{t \to 0} \hat{\rho}_t = \hat{\rho}. \tag{7.91}$$

Now complement the polarity $\hat{\rho}_t$ by its naturally associated polarity $\hat{\pi}_t$. The pair $(\hat{\pi}_t, \hat{\rho}_t)$ then consists of two non-degenerate naturally associated polarities for all $t \neq 0$. (With $\det \hat{\rho}_t$ and $\mu \in \{1, -1\}$ also $\lambda \in \{1, -1\}$ is determined, compare equations (4.281), middle expression.)

The naturally associated polarity to $\hat{\rho}$ is defined by

$$\begin{aligned} \hat{\pi}(P_{\mathbf{b}}) &:= \mathbf{0}, && S(\mathbf{b}) = 0, \\ \hat{\pi}(P_{\mathbf{b}}) &:= \lim_{t \to 0} \left(\frac{\hat{\pi}_t(P_{\mathbf{b}})}{t^{(u^- - 1)}} \right), && S(\mathbf{b}) = 1, \\ \hat{\pi}(A \wedge B) &:= \lambda \cdot \hat{\pi}(A) \vee \hat{\pi}(B) && \forall\, A, B \in \Lambda_n. \end{aligned} \tag{7.92}$$

In the next two subsections we study the quadratic and bilinear forms of naturally associated polarities. In addition, we address the question, whether *both* systems of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$ of an harmonic model of Λ_n can be orthonormal at the same time.

7.6. Pairs of Polarities Representing a Non-Degenerate Quadric

This subsection deals with pairs $(\hat{\pi}, \hat{\rho})$ of naturally associated polarities, which are non-degenerate only.

In these cases, the two quadrics $\mathcal{Q}^{+}_{\hat\pi}$ and $\mathcal{Q}^{-}_{\hat\rho}$, belonging to the pair of polarities $(\hat\pi, \hat\rho)$, coincide, $\mathcal{Q}^{k-}_{\hat\pi} = \mathcal{Q}^{(n-k)+}_{\hat\rho}$. The pair of polarities then represents the different grades k of *one* non-degenerate quadric. This is a consequence of

Theorem 7.16. *Let $\hat\pi$ and $\hat\rho$ form a pair of* non-degenerate *naturally associated polarities $(\hat\pi, \hat\rho)$, where $\hat\pi$ and $\hat\rho$ represent the same transformation in the sense of Theorem 4.39. The quadratic and bilinear forms then are the same in both approaches,*

$$Q^{k+}_{\hat\pi}(X) = Q^{(n-k)-}_{\hat\rho}(X) \qquad \forall\ X \in \Lambda^{k+}_n, \tag{7.93}$$

$$Q^{k-}_{\hat\pi}(X) = Q^{(n-k)+}_{\hat\rho}(X) \qquad \forall\ X \in \Lambda^{k-}_n. \tag{7.94}$$

$$B^{k+}_{\hat\pi}(X,Y) = B^{(n-k)-}_{\hat\rho}(X,Y) \qquad \forall\ X,Y \in \Lambda^{k+}_n, \tag{7.95}$$

$$B^{k-}_{\hat\pi}(X,Y) = B^{(n-k)+}_{\hat\rho}(X,Y) \qquad \forall\ X,Y \in \Lambda^{k-}_n. \tag{7.96}$$

Proof. With the equations

$$\begin{aligned}\Lambda^{k+}_n \ni X &= \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}} = \sum_{S(\mathbf{b})=k} \alpha_{\overline{\mathbf{b}}\mathbf{b}} \mu_{\mathbf{b}} E_{\overline{\mathbf{b}}} \\ &\overset{!}{=} \sum_{S(\mathbf{c})=n-k} \nu_{\mathbf{c}} E_{\mathbf{c}} \in \Lambda^{(n-k)-}_n \quad \Longleftrightarrow \quad \nu_{\mathbf{c}} = \alpha_{\mathbf{c}\overline{\mathbf{c}}} \mu_{\overline{\mathbf{c}}},\end{aligned} \tag{7.97}$$

$$\begin{aligned}\Lambda^{k-}_n \ni X &= \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}} E_{\mathbf{b}} = \sum_{S(\mathbf{b})=k} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \nu_{\mathbf{b}} P_{\overline{\mathbf{b}}} \\ &\overset{!}{=} \sum_{S(\mathbf{c})=n-k} \mu_{\mathbf{c}} P_{\mathbf{c}} \in \Lambda^{(n-k)+}_n \quad \Longleftrightarrow \quad \mu_{\mathbf{c}} = \alpha_{\overline{\mathbf{c}}\mathbf{c}} \nu_{\overline{\mathbf{c}}}\end{aligned} \tag{7.98}$$

we get for all $X = \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}} \in \Lambda^{k+}_n$, for all $\mathbf{d} = \overline{\mathbf{b}}$ and for all $\mathbf{e} = \overline{\mathbf{c}}$ with $S(\mathbf{c}) = k$, $S(\mathbf{d}) = n-k$ and $S(\mathbf{e}) = n-k$

$$\begin{aligned}Q^{k+}_{\hat\pi}(X) &= \sum_{\substack{S(\mathbf{b})=k \\ S(\mathbf{c})=k}} \mu_{\mathbf{b}} \hat\beta_{\mathbf{b}\mathbf{c}} \mu_{\mathbf{c}} = \sum_{\substack{S(\mathbf{b})=k \\ S(\mathbf{c})=k}} \mu_{\mathbf{b}} \alpha_{\overline{\mathbf{b}}\mathbf{b}} \hat\gamma_{\overline{\mathbf{b}}\overline{\mathbf{c}}} \alpha_{\mathbf{c}\overline{\mathbf{c}}} \mu_{\mathbf{c}} \\ &= \sum_{\substack{S(\mathbf{d})=n-k \\ S(\mathbf{e})=n-k}} \mu_{\overline{\mathbf{d}}} \alpha_{\mathbf{d}\overline{\mathbf{d}}} \hat\gamma_{\mathbf{d}\mathbf{e}} \alpha_{\overline{\mathbf{e}}\mathbf{e}} \mu_{\overline{\mathbf{e}}} \\ &= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{d})=n-k \\ S(\mathbf{e})=n-k}} \mu_{\overline{\mathbf{d}}} \alpha_{\mathbf{d}\overline{\mathbf{d}}} \hat\gamma_{\mathbf{d}\mathbf{e}} \alpha_{\mathbf{e}\overline{\mathbf{e}}} \mu_{\overline{\mathbf{e}}} \\ &= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{d})=n-k \\ S(\mathbf{e})=n-k}} \nu_{\mathbf{d}} \hat\gamma_{\mathbf{d}\mathbf{e}} \nu_{\mathbf{e}} = Q^{(n-k)-}_{\hat\rho}(X),\end{aligned} \tag{7.99}$$

for all $X = \sum_{S(\mathbf{b})=k} \nu_\mathbf{b} E_\mathbf{b} \in \Lambda_n^{k-}$, for all $\mathbf{d} = \overline{\mathbf{b}}$ and for all $\mathbf{e} = \overline{\mathbf{c}}$ with $S(\mathbf{c}) = k$, $S(\mathbf{d}) = n-k$ and $S(\mathbf{e}) = n-k$

$$\begin{aligned}
Q_{\hat\rho}^{k-}(X) &= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b})=k\\ S(\mathbf{c})=k}} \nu_\mathbf{b} \hat\gamma_{\mathbf{bc}} \nu_\mathbf{c} \qquad (7.100)\\
&= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{b})=k\\ S(\mathbf{c})=k}} \nu_\mathbf{b} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \hat\beta_{\overline{\mathbf{b}}\overline{\mathbf{c}}} \alpha_{\overline{\mathbf{c}}\mathbf{c}} \nu_\mathbf{c}\\
&= (-1)^{k(n-k)} \sum_{\substack{S(\mathbf{d})=n-k\\ S(\mathbf{e})=n-k}} \nu_{\overline{\mathbf{d}}} \alpha_{\overline{\mathbf{d}}\mathbf{d}} \hat\beta_{\mathbf{de}} \alpha_{\mathbf{e}\overline{\mathbf{e}}} \nu_{\overline{\mathbf{e}}}\\
&= \sum_{\substack{S(\mathbf{d})=n-k\\ S(\mathbf{e})=n-k}} \nu_{\overline{\mathbf{d}}} \alpha_{\overline{\mathbf{d}}\mathbf{d}} \hat\beta_{\mathbf{de}} \alpha_{\overline{\mathbf{e}}\mathbf{e}} \nu_{\overline{\mathbf{e}}} = \sum_{\substack{S(\mathbf{d})=n-k\\ S(\mathbf{e})=n-k}} \mu_\mathbf{d} \hat\beta_{\mathbf{de}} \mu_\mathbf{e}\\
&= Q_{\hat\pi}^{(n-k)+}(X),
\end{aligned}$$

for all $X = \sum_{S(\mathbf{b})=k} \mu_\mathbf{b} P_\mathbf{b} \in \Lambda_n^{k+}$ and for all $Y = \sum_{S(\mathbf{b})=k} \nu_\mathbf{b} P_\mathbf{b} \in \Lambda_n^{k+}$

$$\begin{aligned}
B_{\hat\pi}^{k+}(X,Y) &= \frac{1}{2}\left(Q_{\hat\pi}^{k+}(X+Y) - Q_{\hat\pi}^{k+}(X) - Q_{\hat\pi}^{k+}(Y)\right) \qquad (7.101)\\
&= \frac{1}{2}\left(Q_{\hat\rho}^{(n-k)-}(X+Y) - Q_{\hat\rho}^{(n-k)-}(X) - Q_{\hat\rho}^{(n-k)-}(Y)\right)\\
&= B_{\hat\rho}^{(n-k)-}(X,Y)
\end{aligned}$$

and for all $X = \sum_{S(\mathbf{b})=k} \mu_\mathbf{b} E_\mathbf{b} \in \Lambda_n^{k-}$ and for all $Y = \sum_{S(\mathbf{b})=k} \nu_\mathbf{b} E_\mathbf{b} \in \Lambda_n^{k-}$

$$\begin{aligned}
B_{\hat\rho}^{k-}(X,Y) &= \frac{1}{2}\left(Q_{\hat\rho}^{k-}(X+Y) - Q_{\hat\rho}^{k-}(X) - Q_{\hat\rho}^{k-}(Y)\right) \qquad (7.102)\\
&= \frac{1}{2}\left(Q_{\hat\pi}^{(n-k)+}(X+Y) - Q_{\hat\pi}^{(n-k)+}(X) - Q_{\hat\pi}^{(n-k)+}(Y)\right)\\
&= B_{\hat\pi}^{(n-k)+}(X,Y).
\end{aligned}$$

Thus, equations (7.93), (7.94), (7.95) and (7.96) are true. □

The relations of the following Lemma will be used further on.

Lemma 7.17. *Let $(\hat\pi, \hat\rho)$ represent a pair of two non-null polarities as in Corollary 7.13 with the single difference that both of them are supposed to be non-degenerate. Let $\{P_\mathbf{b}\} \subset \Lambda_n$ represent an orthonormal basis with respect to the quadratic form $Q_{\hat\pi}^{1+}$ and with signature $\mathcal{S}^+ = (s^+, t^+, 0)$.*[14] *And let $\{E'_\mathbf{b}\} \subset \Lambda_n$ represent an orthonormal basis with respect to the quadratic form $Q_{\hat\rho}^{1-}$ and with signature $\mathcal{S}^- = (s^-, t^-, 0)$.*[15] *We then have with $S(\mathbf{u}) = n$*

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[14] Cf. equations (7.65) and (7.66) of Corollary 7.13.

[15] Cf. equations (7.67) and (7.68) of Corollary 7.13.

and $0 \le S(\mathbf{b}) = k \le n$

$$\det \hat{\pi} = \prod_{l=1}^{n} \eta^{+}_{{}_l\mathbf{u}} = (-1)^{t^+}, \tag{7.103}$$

$$\det \hat{\rho} = \prod_{l=1}^{n} \eta^{-}_{{}_l\mathbf{u}} = (-1)^{t^-}, \tag{7.104}$$

$$n = s^+ + t^+ = s^- + t^-, \qquad 0 = u^+ = u^-, \tag{7.105}$$

$$\begin{aligned} \hat{\beta}_{\mathbf{bb}}\hat{\beta}_{\overline{\mathbf{bb}}} &= \lambda^{k-1}\left(\prod_{l=1}^{k} \eta^{+}_{{}_l\mathbf{b}}\right)\lambda^{n-k-1}\left(\prod_{l=1}^{n-k} \eta^{+}_{{}_l\overline{\mathbf{b}}}\right) \qquad (7.106) \\ &= \lambda^{n-2}\det\hat{\pi} = (-1)^{t^+}\lambda^{n-2}, \end{aligned}$$

$$\hat{\gamma}_{\mathbf{bb}}\hat{\gamma}_{\overline{\mathbf{bb}}} = \mu^{n-2}\det\hat{\rho} = (-1)^{t^-}\mu^{n-2}. \tag{7.107}$$

Proof. Equations (7.103) and (7.104) follow from equations (7.73) and (7.74) respectively. Equations (7.106) and (7.107) follow with equations (7.70) and (7.72) respectively. ☐

Now, let us suppose, $(\hat{\pi}, \hat{\rho})$ represents a pair of two non-degenerate and naturally associated polarities with real parameters $\lambda, \mu \in \mathbb{R} \setminus \{0\}$. And let $\{P_{\mathbf{b}}\} \subset \Lambda_n$ be an orthonormal basis with respect to the quadratic form $Q^{1+}_{\hat{\pi}}$ and with signature $\mathcal{S}^+ = (s^+, t^+, 0)$. We want to know, whether or under which conditions the basis $\{E_{\mathbf{b}}\} \subset \Lambda_n$ is orthonormal with respect to the quadratic form $Q^{1-}_{\hat{\rho}}$? The system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$ is supposed to form an harmonic model of projective algebra Λ_n according to Definition 4.26. Thus, let us compute the quadratic and bilinear forms with respect to the basis $\{E_{\mathbf{b}}\}$ for $S(\mathbf{b}) = S(\mathbf{c}) = 1$ and $\mathbf{b} \neq \mathbf{c}$,

$$\begin{aligned} Q^{1-}_{\hat{\rho}}(E_{\mathbf{b}}) &= Q^{(n-1)+}_{\hat{\pi}}(E_{\mathbf{b}}) = Q^{(n-1)+}_{\hat{\pi}}(\alpha_{\mathbf{b}\overline{\mathbf{b}}}P_{\overline{\mathbf{b}}}) = Q^{(n-1)+}_{\hat{\pi}}(P_{\overline{\mathbf{b}}}) \qquad (7.108) \\ &= \hat{\beta}_{\overline{\mathbf{bb}}} = \frac{\lambda^{n-2}\det\hat{\pi}}{\hat{\beta}_{\mathbf{bb}}} = \frac{\lambda^{n-2}\det\hat{\pi}}{\eta^{+}_{\mathbf{b}}}, \end{aligned}$$

$$\begin{aligned} B^{1-}_{\hat{\rho}}(E_{\mathbf{b}}, E_{\mathbf{c}}) &= B^{(n-1)+}_{\hat{\pi}}(E_{\mathbf{b}}, E_{\mathbf{c}}) = B^{(n-1)+}_{\hat{\pi}}(\alpha_{\mathbf{b}\overline{\mathbf{b}}}P_{\overline{\mathbf{b}}}, \alpha_{\mathbf{c}\overline{\mathbf{c}}}P_{\overline{\mathbf{c}}}) \qquad (7.109) \\ &= \alpha_{\mathbf{b}\overline{\mathbf{b}}}\alpha_{\mathbf{c}\overline{\mathbf{c}}} \cdot B^{(n-1)+}_{\hat{\pi}}(P_{\overline{\mathbf{b}}}, P_{\overline{\mathbf{c}}}) = 0. \end{aligned}$$

The basis $\{E_{\mathbf{b}}\}$ is orthonormal with respect to the quadratic form $Q^{1-}_{\hat{\rho}}$, if and only if $\lambda = \pm 1$. And with the left side of equations (4.281) it follows also $\mu = \pm 1$.

In a similar way we could have started with the orthonormal basis $\{E_{\mathbf{b}}\} \subset \Lambda_n$ and would have come to the conclusion, the basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$ is orthogonal too, if $\mu = \pm 1$ and consequently $\lambda = \pm 1$.

So with

$$\eta^{+}_{\mathbf{b}}\eta^{-}_{\mathbf{b}} = \lambda^{n-2}\det\hat{\pi} = \mu^{n-2}\det\hat{\rho} = \frac{1}{\lambda\mu}, \tag{7.110}$$

$$\det\hat{\pi} = \frac{1}{\mu\,\lambda^{n-1}}, \quad \det\hat{\rho} = \frac{1}{\lambda\,\mu^{n-1}}, \quad \frac{\det\hat{\pi}}{\det\hat{\rho}} = \left(\frac{\mu}{\lambda}\right)^{n-2} \tag{7.111}$$

from equations (7.108) and (4.281) we get an overview on the different cases. See Table 9.

The equivalence relation for multi vectors in projective geometry $\mathcal{P}_n$ from Definition 5.2 implies, that $\hat{\pi}$ and $\hat{\pi}_\xi$ as well as $\hat{\rho}$ and $\hat{\rho}_o$ with

$$\begin{aligned} &\hat{\pi}_\xi(A_{\overline{k}}) := \xi^k \cdot \hat{\pi}(A_{\overline{k}}), && \hat{\rho}_o(A_{\overline{k}}) := o^k \cdot \hat{\rho}(A_{\overline{k}}), \\ &A_{\overline{k}} \in \Lambda_n^k, && \xi, o \in \mathbb{F} \setminus \{0\} \end{aligned} \tag{7.112}$$

represent each the same non-null polarity. Nevertheless, the determinants are changing by the factors ξ^n and o^n respectively,

$$\det(\hat{\pi}_\xi) = \xi^n \det \hat{\pi}, \qquad \det(\hat{\rho}_o) = o^n \det \hat{\rho}. \tag{7.113}$$

In the case of real factors $\xi, o \in \mathbb{R} \setminus \{0\}$ positive and negative determinants are separated for even dimensions n and not separated for odd dimensions n. Since the sign of the determinant represents the orientation of the respective projective space, it is said, that projective geometry $\mathcal{P}_n$ of even dimensions n can be oriented — $\det \hat{\pi} = 1$ and $\det \hat{\pi} = -1$ are separated — and projective geometry $\mathcal{P}_n$ of odd dimensions n not — $\det \hat{\pi} = 1$ and $\det \hat{\pi} = -1$ are not separated.

In projective geometry $\mathcal{P}_n$, a non-degenerate but otherwise generic pair of polarities $(\hat{\pi}, \hat{\rho})$ and the pair of polarities $(\hat{\pi}_\xi, \hat{\rho}_o)$ is the same. Eventually the two non-degenerate quadrics $\mathcal{Q}_{\hat{\pi}}$ and $\mathcal{Q}_{\hat{\rho}}$ may coincide, e. g. in case of a naturally associated pair $(\hat{\pi}, \hat{\rho})$. In case of even dimensions n, the signs of the respective determinants are the same, i. e. $\operatorname{sign} \det \hat{\pi} = \operatorname{sign} \det \hat{\pi}_\xi$ and $\operatorname{sign} \det \hat{\rho} = \operatorname{sign} \det \hat{\rho}_o$; in case of odd dimensions n the signs of the determinants $\det \hat{\pi}_\xi$ and $\det \hat{\rho}_o$ depend on the factors $\xi, o \in \mathbb{R} \setminus \{0\}$ and can differ independently from the signs of $\det \hat{\pi}$ and $\det \hat{\rho}$ respectively.

In the third column of Table 9 the pair $(\det \hat{\pi}, \det \hat{\rho})$ is listed and can take on, in the context of non-degenerate naturally associated pairs of polarities $(\hat{\pi}, \hat{\rho})$ with respect to an harmonic orthogonal system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$, one of the four possible values: $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$. We call the combinations $(\det \hat{\pi}, \det \hat{\rho})$ lying on the first angle bisector *even* and the combinations $(\det \hat{\pi}, \det \hat{\rho})$ lying on the second angle bisector *odd*,

$$\left. \begin{aligned} &(\det \hat{\pi}, \det \hat{\rho}) \text{ even} && :\Longleftrightarrow && \operatorname{sign} \det \hat{\pi} = \operatorname{sign} \det \hat{\rho}, \\ &(\det \hat{\pi}, \det \hat{\rho}) \text{ odd} && :\Longleftrightarrow && \operatorname{sign} \det \hat{\pi} = -\operatorname{sign} \det \hat{\rho}. \end{aligned} \right\} \tag{7.114}$$

From Table 9 we see, in the case of even dimensions n there are only *even* combinations of naturally associated polarities. Compare also the right side of equations (7.111). And since the pairs (e1) $(\det \hat{\pi}, \det \hat{\rho}) = (1, 1)$ and (e2) $(\det \hat{\pi}, \det \hat{\rho}) = (-1, -1)$ are separated, they represent different quadrics.

In the case of odd dimensions n, no two pairs $(\det \hat{\pi}, \det \hat{\rho})$ are separated anymore.

We summarise the result of this subsection in

Definition and Theorem 7.18 (Harmonic Orthonormal System of Bases I). Let $(\hat{\pi}, \hat{\rho})$ represent a pair of two non-degenerate naturally associated polarities according to Definition 7.15 with $\lambda, \mu \in \{1, -1\}$. Then there exists a

n	λ, μ	$\det\hat\pi$, $\det\hat\rho$	Signatures $\mathcal{S}^{\pm}$
even	$\lambda = \mu = 1$	(e1) $\det\hat\pi = \det\hat\rho = 1$	$\eta_{\mathbf{b}}^{+} = \eta_{\mathbf{b}}^{-}$ $s := s^{+} = s^{-}$ even $t := t^{+} = t^{-}$ even $(s, t, 0)^{+}$, $(s, t, 0)^{-}$
	$\lambda = \mu = -1$	(e1) $\det\hat\pi = \det\hat\rho = 1$	$\eta_{\mathbf{b}}^{+} = \eta_{\mathbf{b}}^{-}$ $s := s^{+} = s^{-}$ even $t := t^{+} = t^{-}$ even $(s, t, 0)^{+}$, $(s, t, 0)^{-}$
	$\lambda = -\mu = 1$	(e2) $\det\hat\pi = \det\hat\rho = -1$	$\eta_{\mathbf{b}}^{+} = -\eta_{\mathbf{b}}^{-}$ $v := s^{+} = t^{-}$ odd $w := t^{+} = s^{-}$ odd $(v, w, 0)^{+}$, $(w, v, 0)^{-}$
	$\lambda = -\mu = -1$	(e2) $\det\hat\pi = \det\hat\rho = -1$	$\eta_{\mathbf{b}}^{+} = -\eta_{\mathbf{b}}^{-}$ $v := s^{+} = t^{-}$ odd $w := t^{+} = s^{-}$ odd $(v, w, 0)^{+}$, $(w, v, 0)^{-}$
odd	$\lambda = \mu = 1$	(o1) $\det\hat\pi = \det\hat\rho = 1$	$\eta_{\mathbf{b}}^{+} = \eta_{\mathbf{b}}^{-}$ $s := s^{+} = s^{-}$ odd $t := t^{+} = t^{-}$ even $(s, t, 0)^{+}$, $(s, t, 0)^{-}$
	$\lambda = \mu = -1$	(o1) $\det\hat\pi = \det\hat\rho = -1$	$\eta_{\mathbf{b}}^{+} = \eta_{\mathbf{b}}^{-}$ $s := s^{+} = s^{-}$ even $t := t^{+} = t^{-}$ odd $(s, t, 0)^{+}$, $(s, t, 0)^{-}$
	$\lambda = -\mu = 1$	(o2) $\det\hat\pi = -\det\hat\rho = -1$	$\eta_{\mathbf{b}}^{+} = -\eta_{\mathbf{b}}^{-}$ $v := s^{+} = t^{-}$ even $w := t^{+} = s^{-}$ odd $(v, w, 0)^{+}$, $(w, v, 0)^{-}$
	$\lambda = -\mu = -1$	(o2) $\det\hat\pi = -\det\hat\rho = 1$	$\eta_{\mathbf{b}}^{+} = -\eta_{\mathbf{b}}^{-}$ $v := s^{+} = t^{-}$ odd $w := t^{+} = s^{-}$ even $(v, w, 0)^{+}$, $(w, v, 0)^{-}$

TABLE 9. Signatures in space and counterspace of a non-degenerate pair $(\hat\pi, \hat\rho)$ of naturally associated polarities with respect to an harmonic orthogonal system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$ in $\mathcal{P}_n$.

system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$, which forms an harmonic model of projective algebra Λ_n according to Definition 4.26 and which is at the same time orthonormal with respect to the quadratic forms $Q_{\hat\pi}^{1+}$ and $Q_{\hat\rho}^{1-}$ and the bilinear forms $B^{1+}_{\hat{\pi}}$ and $B^{1-}_{\hat{\rho}}$ respectively with

$$Q^{1+}_{\hat{\pi}}(P_{\mathbf{b}}) \cdot Q^{1-}_{\hat{\rho}}(E_{\mathbf{b}}) = \eta^{+}_{\mathbf{b}} \eta^{-}_{\mathbf{b}} = \frac{1}{\lambda\mu}, \tag{7.115}$$

$$\det \hat{\pi} = \frac{1}{\mu\,\lambda^{n-1}}, \qquad \det \hat{\rho} = \frac{1}{\lambda\,\mu^{n-1}}, \qquad \frac{\det \hat{\pi}}{\det \hat{\rho}} = \left(\frac{\mu}{\lambda}\right)^{n-2}. \tag{7.116}$$

A list of all different cases is displayed in Table 9.

We call the above described system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$ an *harmonic orthonormal system of bases* with respect to the pair $(\hat{\pi}, \hat{\rho})$ of two non-degenerate naturally associated polarities. The term *harmonic orthonormal system of bases* is also used with respect to a pair $(\hat{\pi}, \hat{\rho})$ of two degenerate naturally associated polarities.

Proof. The existence of an harmonic orthonormal system of bases for a pair $(\hat{\pi}, \hat{\rho})$ of two degenerate naturally associated polarities is described in Theorem 7.19 and 7.22 of the next subsection. For everything else see above. ☐

7.7. Transition to Pairs of Two Degenerate Naturally Associated Polarities

This subsection makes the transition to pairs $(\hat{\pi}, \hat{\rho})$ of naturally associated polarities, which are degenerate and non-vanishing. We thereby follow the procedure proposed in Definition 7.15 as a preliminary version, how to define and construct degenerate naturally associated pairs of polarities $(\hat{\pi}, \hat{\rho})$. In this procedure, we parameterise non-degenerate pairs of polarities $(\hat{\pi}_t, \hat{\rho}_t)$ and their quadrics $\mathcal{Q}^{(t)\pm}_{(\hat{\pi}_t, \hat{\rho}_t)}$. In the limit $t \to 0$ the pairs of non-degenerate polarities as well as their non-degenerate quadrics degenerate to the pair $(\hat{\pi}, \hat{\rho})$ of two degenerate naturally associated polarities as well as to two degenerate quadrics $(\mathcal{Q}^{+}_{\hat{\pi}}, \mathcal{Q}^{-}_{\hat{\rho}})$.

Let us construct now in detail the degenerate naturally associated pairs of polarities $(\hat{\pi}, \hat{\rho})$.

In the first case, let a generic, but degenerate polarity $\hat{\pi}$ from the plus to the minus approach be given (including a fixed $\lambda \in \{1, -1\}$) with respect to an orthonormal basis $\{P_{\mathbf{b}}\}$ of signature S^{+} and with quadratic form $Q^{1+}_{\hat{\pi}}$ according to the equations (7.79), (7.80) and (7.81) of Definition 7.15.

Our choice for the in general non-degenerate polarity $\hat{\pi}_t$ is displayed in equations (7.85), and (7.86) of Definition 7.15. This choice still comes with the flexibility of adjusting the number m^{+}_{1} of parameters $-t$ with a negative sign and the number m^{+}_{2} of parameters t with a positive sign within the limits of equation (7.86). And $\hat{\pi}_t$ was constructed in such a way, that it degenerates to $\hat{\pi}$ according to equation (7.87) of Definition 7.15.

For each non-degenerate polarity $\hat{\pi}_t$ (with $t \neq 0$), there is a naturally associated polarity $\hat{\rho}_t$ according to Definition 7.15 and Theorem 4.39. The degenerate and naturally associated polarity $\hat{\rho}$ to $\hat{\pi}$ is displayed in equations (7.88) of Definition 7.15.

We determine now the polarity $\hat{\rho}$, its matrix $\hat{\Gamma}_{\bar{1}}$, its quadratic form $Q^{1-}_{\hat{\rho}}$, its parameter μ and the corresponding signature S^{-}. In order to do so, let

$S(\mathbf{b}) = 1$. For $t \neq 0$, the non-degenerate polarity $\hat{\rho}_t$, evaluated over the 1-vectors $E_{\mathbf{b}}$ of the minus approach, returns 1-vectors in the plus approach,

$$\begin{aligned}\hat{\rho}_t(E_{\mathbf{b}}) &= \sum_{S(\mathbf{c})=1} \hat{\gamma}^{(t)}_{\mathbf{bc}} P_{\mathbf{c}} = \sum_{S(\mathbf{c})=1} \alpha_{\mathbf{b}\overline{\mathbf{b}}} \hat{\beta}^{(t)}_{\overline{\mathbf{b}}\overline{\mathbf{c}}} \alpha_{\overline{\mathbf{c}}\mathbf{c}} P_{\mathbf{c}} \\ &= \left(\sum_{S(\mathbf{b})=1} (-1)^{n-1} \lambda^{n-2} \prod_{l=1}^{n-1} \hat{\beta}^{(t)}_{{}_l\overline{\mathbf{b}}\,{}_l\overline{\mathbf{b}}} P_{\mathbf{b}} \right) \delta_{\mathbf{bc}} \\ &= \left((-1)^{n-1} \lambda^{n-2} \sum_{S(\mathbf{b})=1} \prod_{l=1}^{n-1} \varepsilon^{(t)+}_{{}_l\overline{\mathbf{b}}} P_{\mathbf{b}} \right) \delta_{\mathbf{bc}}\end{aligned} \tag{7.117}$$

where we used equation (4.279) from Theorem 4.39, equation (4.148) from Theorem 4.26, equation (4.83) from Definition and Theorem 4.15 and equation (7.85) from Definition 7.15. In order to determine the parameter $\mu \neq 0$, we use the left expression from equations (4.281) and $\hat{\pi}_{t=1}$,

$$\mu = \frac{1}{\lambda^{n-1} \det \hat{\pi}_{t=1}} = \frac{1}{\lambda^{n-1} (-1)^{t^+ + m_1^+}}. \tag{7.118}$$

Taking the limit $t \to 0$ according to equation (7.87) from Definition 7.15, we get the matrix $\hat{\Gamma}_{\overline{1}}$ of the polarity $\hat{\rho}$,

$$\hat{\gamma}_{\mathbf{bc}} = \lim_{t \to 0} \left(\frac{\hat{\gamma}^{(t)}_{\mathbf{bc}}}{t^{(u^+ - 1)}} \right) = \begin{cases} 0, & \mathbf{b} \neq \mathbf{c}, \\ (-1)^{n-1} \eta^-_{\mathbf{b}}, & \mathbf{b} = \mathbf{c}, \end{cases} \tag{7.119}$$

with $S(\mathbf{u}) = n$,

$$\eta^-_{{}_l\mathbf{u}} = \begin{cases} 0, & 1 \leq l \leq u^-, \\ \lambda^{n-2} (-1)^{t^+ + m_1^+ - 1}, & u^- + 1 \leq l \leq u^- + m_1^+, \\ \lambda^{n-2} (-1)^{t^+ + m_1^+}, & u^- + m_1^+ + 1 \leq l \leq n, \end{cases} \tag{7.120}$$

and

$$u^- = s^+ + t^+, \tag{7.121}$$

$$s^- + t^- = m_1^+ + m_2^+, \tag{7.122}$$

$$s^- := \begin{cases} m_2^+, & \text{in case } \lambda^{n-2}(-1)^{t^+ + m_1^+} = 1 \text{ and } \lambda\mu = 1, \\ m_1^+, & \text{in case } \lambda^{n-2}(-1)^{t^+ + m_1^+} = -1 \text{ and } \lambda\mu = -1, \end{cases} \tag{7.123}$$

$$t^- := \begin{cases} m_1^+, & \text{in case } \lambda^{n-2}(-1)^{t^+ + m_1^+} = 1 \text{ and } \lambda\mu = 1, \\ m_2^+, & \text{in case } \lambda^{n-2}(-1)^{t^+ + m_1^+} = -1 \text{ and } \lambda\mu = -1. \end{cases} \tag{7.124}$$

Please note the following detail in equations (7.123) and (7.124). With equation (7.118) the case $\lambda^{n-2}(-1)^{t^+ + m_1^+} = 1$ implies $\lambda\mu = 1$ and the complementing case $\lambda^{n-2}(-1)^{t^+ + m_1^+} = -1$ implies $\lambda\mu = -1$.

The quadratic form with respect to the degenerate polarity $\hat{\rho}$ then reads

$$Q^{1-}_{\hat{\rho}}(E_{{}_l\mathbf{u}}) = \eta^-_{{}_l\mathbf{u}}, \qquad 1 \leq l \leq n, \tag{7.125}$$

and its signature

$$\mathcal{S}^{-} = (s^{-}, t^{-}, u^{-}), \qquad 1 \leq u^{-} < n, \tag{7.126}$$

is fixed by equations (7.121), (7.123) and (7.124).

Without loss of generality, in case of $\lambda\mu = -1$ we may reorder the indices of $P_{{}_{l}\mathbf{u}}$ for $s^{+} + t^{+} + 1 \leq l \leq n$ such, that equations (7.80) and (7.81) remain valid and we get instead of equation (7.120) the following expression for $\lambda\mu = 1$ and $\lambda\mu = -1$,

$$\eta^{-}_{{}_{l}\mathbf{u}} = \begin{cases} 0, & 1 \leq l \leq u^{-}, \\ -1, & u^{-} + 1 \leq l \leq u^{-} + t^{-}, \\ 1, & u^{-} + t^{-} + 1 \leq l \leq u^{-} + t^{-} + s^{-} = n, \end{cases} \tag{7.127}$$

where the parameters of the signature S^{-} are still fixed by equations (7.121), (7.123) and (7.124) and we can choose for each pair (s^{-}, t^{-})

$$\mu = \pm\frac{1}{\lambda}. \tag{7.128}$$

In the second and complementing case, let a generic, but degenerate polarity $\hat{\rho}$ from the minus to the plus approach be given (including a fixed $\mu \in \{1, -1\}$) with respect to an orthonormal basis $\{E_{\mathbf{b}}\}$ of signature S^{-} and with quadratic form $Q^{1-}_{\hat{\rho}}$ according to the equations (7.82), (7.83) and (7.84) of Definition 7.15.

Our choice for the in general non-degenerate polarity $\hat{\rho}_t$ is displayed in equations (7.89) and (7.90) of Definition 7.15. This choice still comes with the flexibility of adjusting the number m^{-}_{1} of parameters $-t$ with a negative sign and the number m^{-}_{2} of parameters t with a positive sign within the limits of equation (7.90). And $\hat{\rho}_t$ was constructed in such a way, that it degenerates to $\hat{\rho}$ according to equation (7.91) of Definition 7.15.

For each non-degenerate polarity $\hat{\rho}_t$ (with $t \neq 0$), there is a naturally associated polarity $\hat{\pi}_t$ according to Definition 7.15 and Theorem 4.39. The degenerate and naturally associated polarity $\hat{\pi}$ to $\hat{\rho}$ is displayed in equations (7.92) of Definition 7.15.

We determine now the polarity $\hat{\pi}$, its matrix $\hat{B}_{\bar{1}}$, its quadratic form $Q^{1+}_{\hat{\pi}}$, its parameter λ and the corresponding signature S^{+}. In order to do so, let $S(\mathbf{b}) = 1$. For $t \neq 0$, the non-degenerate polarity $\hat{\pi}_t$, evaluated over the 1-vectors $P_{\mathbf{b}}$ of the plus approach, returns 1-vectors in the minus approach,

$$\begin{aligned} \hat{\pi}_t(P_{\mathbf{b}}) &= \sum_{S(\mathbf{c})=1} \hat{\beta}^{(t)}_{\mathbf{bc}} E_{\mathbf{c}} = \sum_{S(\mathbf{c})=1} \alpha_{\bar{\mathbf{b}}\mathbf{b}} \hat{\gamma}^{(t)}_{\bar{\mathbf{b}}\bar{\mathbf{c}}} \alpha_{\mathbf{c}\bar{\mathbf{c}}} E_{\mathbf{c}} \\ &= \left(\sum_{S(\mathbf{b})=1} (-1)^{n-1} \mu^{n-2} \prod_{l=1}^{n-1} \hat{\gamma}^{(t)}_{{}_{l}\bar{\mathbf{b}}\,{}_{l}\bar{\mathbf{b}}} E_{\mathbf{b}} \right) \delta_{\mathbf{bc}} \\ &= \left(\mu^{n-2} \sum_{S(\mathbf{b})=1} \prod_{l=1}^{n-1} \varepsilon^{(t)-}_{{}_{l}\bar{\mathbf{b}}} E_{\mathbf{b}} \right) \delta_{\mathbf{bc}} \end{aligned} \tag{7.129}$$

where we used equation (4.280) from Theorem 4.39, equation (4.148) from Theorem 4.26, equation (4.87) from Definition and Theorem 4.15 and equation (7.89) from Definition 7.15. In order to determine the parameter $\lambda \neq 0$, we use the middle expression from equations (4.281) and $\hat{\rho}_{t=1}$,

$$\lambda = \frac{1}{\mu^{n-1} \det \hat{\rho}_{t=1}} = \frac{1}{\mu^{n-1}(-1)^{t^- + m_1^-}}. \tag{7.130}$$

Taking the limit $t \to 0$ according to equation (7.92) from Definition 7.15, we get the matrix $\hat{B}_{\bar{1}}$ of the polarity $\hat{\pi}$,

$$\hat{\beta}_{\mathbf{bc}} = \lim_{t \to 0} \left(\frac{\hat{\beta}^{(t)}_{\mathbf{bc}}}{t^{(u^- - 1)}} \right) = \begin{cases} 0, & \mathbf{b} \neq \mathbf{c}, \\ \eta^{+}_{\mathbf{b}}, & \mathbf{b} = \mathbf{c}, \end{cases} \tag{7.131}$$

with $S(\mathbf{u}) = n$,

$$\eta^{+}_{{}_l\mathbf{u}} = \begin{cases} 0, & 1 \leq l \leq u^+, \\ \mu^{n-2}(-1)^{t^- + m_1^- - 1}, & u^+ + 1 \leq l \leq u^+ + m_1^-, \\ \mu^{n-2}(-1)^{t^- + m_1^-}, & u^+ + m_1^- + 1 \leq l \leq n, \end{cases} \tag{7.132}$$

and

$$u^+ = s^- + t^-, \tag{7.133}$$

$$s^+ + t^+ = m_1^- + m_2^-, \tag{7.134}$$

$$s^+ := \begin{cases} m_2^-, & \text{in case } \mu^{n-2}(-1)^{t^- + m_1^-} = 1 \text{ and } \lambda\mu = 1, \\ m_1^-, & \text{in case } \mu^{n-2}(-1)^{t^- + m_1^-} = -1 \text{ and } \lambda\mu = -1, \end{cases} \tag{7.135}$$

$$t^+ := \begin{cases} m_1^-, & \text{in case } \mu^{n-2}(-1)^{t^- + m_1^-} = 1 \text{ and } \lambda\mu = 1, \\ m_2^-, & \text{in case } \mu^{n-2}(-1)^{t^- + m_1^-} = -1 \text{ and } \lambda\mu = -1. \end{cases} \tag{7.136}$$

Please note the following detail in equations (7.135) and (7.136). With equation (7.130) the case $\mu^{n-2}(-1)^{t^- + m_1^-} = 1$ implies $\lambda\mu = 1$ and the complementing case $\mu^{n-2}(-1)^{t^- + m_1^-} = -1$ implies $\lambda\mu = -1$.

The quadratic form with respect to the degenerate polarity $\hat{\pi}$ then reads

$$Q^{1+}_{\hat{\pi}}(P_{{}_l\mathbf{u}}) = \eta^{+}_{{}_l\mathbf{u}}, \qquad 1 \leq l \leq n, \tag{7.137}$$

and its signature

$$\mathcal{S}^+ = (s^+, t^+, u^+), \qquad 1 \leq u^+ < n, \tag{7.138}$$

is fixed by equations (7.133), (7.135) and (7.136).

Without loss of generality, in case of $\lambda\mu = -1$ we may reorder the indices of $E_{{}_l\mathbf{u}}$ for $s^- + t^- + 1 \leq l \leq n$ such, that equations (7.83) and (7.84) remain valid and we get instead of equation (7.132) the following expression for $\lambda\mu = 1$ and $\lambda\mu = -1$,

$$\eta^{+}_{{}_l\mathbf{u}} = \begin{cases} 0, & 1 \leq l \leq u^+, \\ -1, & u^+ + 1 \leq l \leq u^+ + t^+, \\ 1, & u^+ + t^+ + 1 \leq l \leq u^+ + t^+ + s^+ = n, \end{cases} \tag{7.139}$$

where the parameters of the signature S^+ are still fixed by equations (7.133), (7.135) and (7.136) and we can choose for each pair (s^+, t^+)

$$\mu = \pm\frac{1}{\lambda}. \tag{7.140}$$

We summarize the result of this subsection in

Theorem 7.19 (Harmonic Orthonormal System of Bases II. Preliminary Version). *Let $(\hat{\pi}, \hat{\rho})$ represent a pair of two degenerate naturally associated polarities according to Definition 7.15 with $\lambda, \mu \in \{1, -1\}$. Then there exists a system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$, which forms an harmonic model of projective algebra Λ_n according to Definition 4.26 and is at the same time orthonormal with respect to the quadratic forms $Q^{1+}_{\hat{\pi}}$ and $Q^{1-}_{\hat{\rho}}$ and the bilinear forms $B^{1+}_{\hat{\pi}}$ and $B^{1-}_{\hat{\rho}}$ respectively. In detail we have:*

(C1) If the degenerate polarity $\hat{\pi}$ is given (including a fixed λ, i. e. $\lambda = 1$ or $\lambda = -1$), represent $\hat{\pi}$ with respect to the basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$ with signature

$$\mathcal{S}^+ = (s^+, t^+, u^+), \qquad 1 \leq u^+ < n, \tag{7.141}$$

i. e.

$$Q^{1+}_{\hat{\pi}}(P_{{}_l\mathbf{u}}) = \hat{\beta}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^+_{{}_l\mathbf{u}}, \qquad S(\mathbf{u}) = n, \quad 1 \leq l \leq n, \tag{7.142}$$

with

$$\begin{aligned} \eta^+_{{}_l\mathbf{u}} &= 1, & 1 &\leq l \leq s^+, \\ \eta^+_{{}_l\mathbf{u}} &= -1, & s^+ + 1 &\leq l \leq s^+ + t^+, \\ \eta^+_{{}_l\mathbf{u}} &= 0, & s^+ + t^+ + 1 &\leq l \leq s^+ + t^+ + u^+ = n. \end{aligned} \tag{7.143}$$

then one of the following degenerate polarities $\hat{\rho}$, depending on the choice of parameter s^- and t^-, with respect to the basis $\{E_{\mathbf{b}}\} \subset \Lambda_n$ with signature

$$\mathcal{S}^- = (s^-, t^-, u^-), \qquad 1 \leq u^- < n, \tag{7.144}$$

represents the to $\hat{\pi}$ naturally associated polarity $\hat{\rho}$ and is orthonormal with respect to the basis $\{E_{\mathbf{b}}\} \subset \Lambda_n$,

$$Q^{1-}_{\hat{\rho}}(E_{{}_l\mathbf{u}}) = (-1)^{n-1}\hat{\gamma}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^-_{{}_l\mathbf{u}}, \quad S(\mathbf{u}) = n, \quad 1 \leq l \leq n, \tag{7.145}$$

with

$$\eta^-_{{}_l\mathbf{u}} = \begin{cases} 0, & 1 \leq l \leq u^-, \\ -1, & u^- + 1 \leq l \leq u^- + t^-, \\ 1, & u^- + t^- + 1 \leq l \leq u^- + t^- + s^- = n, \end{cases} \tag{7.146}$$

where

$$u^- = s^+ + t^+, \tag{7.147}$$

$$s^- + t^- = u^+, \qquad 0 \leq s^-, t^- \leq u^+, \tag{7.148}$$

$$\mu = \pm\frac{1}{\lambda}. \tag{7.149}$$

(C2) If the degenerate polarity $\hat{\rho}$ is given (including a fixed μ, i. e. $\mu = 1$ or $\mu = -1$), represent $\hat{\rho}$ with respect to the basis $\{E_{\mathbf{b}}\} \subset \Lambda_n$ with signature

$$\mathcal{S}^- = (s^-, t^-, u^-), \qquad 1 \leq u^- < n, \tag{7.150}$$

i. e.

$$Q^{1-}_{\hat{\rho}}(E_{{}_l\mathbf{u}}) = (-1)^{n-1}\hat{\gamma}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^-_{{}_l\mathbf{u}}, \quad S(\mathbf{u}) = n, \quad 1 \leq l \leq n, \tag{7.151}$$

with

$$\begin{aligned} \eta^-_{{}_l\mathbf{u}} &= 1, & & 1 \leq l \leq s^-, \\ \eta^-_{{}_l\mathbf{u}} &= -1, & & s^- + 1 \leq l \leq s^- + t^-, \\ \eta^-_{{}_l\mathbf{u}} &= 0, & & s^- + t^- + 1 \leq l \leq s^- + t^- + u^- = n. \end{aligned} \tag{7.152}$$

then one of the following degenerate polarities $\hat{\pi}$, depending on the choice of parameter s^+ and t^+, with respect to the basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$ with signature

$$\mathcal{S}^+ = (s^+, t^+, u^+), \qquad 1 \leq u^+ < n, \tag{7.153}$$

represents the to $\hat{\rho}$ naturally associated polarity $\hat{\pi}$ and is orthonormal with respect to the basis $\{P_{\mathbf{b}}\} \subset \Lambda_n$,

$$Q^{1+}_{\hat{\pi}}(P_{{}_l\mathbf{u}}) = \hat{\beta}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^+_{{}_l\mathbf{u}}, \qquad S(\mathbf{u}) = n, \quad 1 \leq l \leq n, \tag{7.154}$$

with

$$\eta^+_{{}_l\mathbf{u}} = \begin{cases} 0, & 1 \leq l \leq u^+, \\ -1, & u^+ + 1 \leq l \leq u^+ + t^+, \\ 1, & u^+ + t^+ + 1 \leq l \leq u^+ + t^+ + s^+ = n, \end{cases} \tag{7.155}$$

where

$$u^+ = s^- + t^-, \tag{7.156}$$

$$s^+ + t^+ = u^-, \qquad 0 \leq s^+, t^+ \leq u^-, \tag{7.157}$$

$$\lambda = \pm\frac{1}{\mu}. \tag{7.158}$$

Proof. See the considerations in this subsection above. □

The parameters λ and μ can be chosen arbitrarily from $\{1, -1\}$ for any combination of signatures S^+ and S^-. See equations (7.158) and (7.149). This is why we do not need to mention them in the cases of degenerate pairs of polarities $(\hat{\pi}, \hat{\rho})$ anymore, since they can be applied arbitrarily anyway.

Now, on the basis of the following theorem, we will extend and simplify the definition of what degenerate naturally associated pairs of polarities are.

Theorem 7.20. *Let $(\hat{\pi}, \hat{\rho})$ form a pair of two degenerate naturally associated polarities according to Definition 7.15. We then have*

$$\hat{\pi}\hat{\rho}(X) = \hat{\rho}\hat{\pi}(X) = \mathbf{0} \qquad\qquad \forall\ X \in \Lambda_n. \tag{7.159}$$

Proof. We prove equation (7.159) for the basis-1-vectors $P_{\mathbf{b}} = \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}}$ and $E_{\mathbf{b}} = \alpha_{\mathbf{b}\overline{\mathbf{b}}} P_{\overline{\mathbf{b}}}$ with $S(\mathbf{b}) = 1$,

$$\hat{\pi}\hat{\rho}(E_{\mathbf{b}}) = (-1)^{n-1}\eta^{-}_{\mathbf{b}} \cdot \hat{\pi}(P_{\mathbf{b}}) = (-1)^{n-1}\eta^{-}_{\mathbf{b}}\eta^{+}_{\mathbf{b}} \cdot E_{\mathbf{b}} = \mathbf{0}, \tag{7.160}$$

$$\begin{aligned} \hat{\rho}\hat{\pi}(E_{\mathbf{b}}) &= \alpha_{\mathbf{b}\overline{\mathbf{b}}} \cdot \hat{\rho}\hat{\pi}(P_{\overline{\mathbf{b}}}) \\ &= (-1)^{(n-1)^2}(\lambda\mu)^{n-2}\left(\prod_{l=1}^{n-1}\eta^{+}_{{}_l\mathbf{b}}\eta^{-}_{{}_l\mathbf{b}}\right) \cdot E_{\mathbf{b}} = \mathbf{0}. \end{aligned} \tag{7.161}$$

$$\hat{\rho}\hat{\pi}(P_{\mathbf{b}}) = \eta^{+}_{\mathbf{b}} \cdot \hat{\rho}(E_{\mathbf{b}}) = (-1)^{n-1}\eta^{+}_{\mathbf{b}}\eta^{-}_{\mathbf{b}} \cdot P_{\mathbf{b}} = \mathbf{0}. \tag{7.162}$$

$$\begin{aligned} \hat{\pi}\hat{\rho}(P_{\mathbf{b}}) &= \alpha_{\overline{\mathbf{b}}\mathbf{b}} \cdot \hat{\pi}\hat{\rho}(E_{\overline{\mathbf{b}}}) \\ &= (-1)^{(n-1)^2}(\lambda\mu)^{n-2}\left(\prod_{l=1}^{n-1}\eta^{-}_{{}_l\mathbf{b}}\eta^{+}_{{}_l\mathbf{b}}\right) \cdot P_{\mathbf{b}} = \mathbf{0}. \end{aligned} \tag{7.163}$$

As a consequence, equation (7.159) is true for all $X \in \Lambda_n$. □

According to Theorem 4.39, in case of non-degenerate polarities, there are always the two odd algebra isomorphisms $\hat{\pi}$ and $\hat{\rho}$ representing the same polarity, i. e., we have a pair of naturally associated polarities $(\hat{\pi}, \hat{\rho})$ with

$$\hat{\pi}^2(X) = \hat{\pi}\hat{\rho}(X) = \hat{\rho}\hat{\pi}(X) = \hat{\rho}^2(X) = \varepsilon_k X \tag{7.164}$$

for all $X \in \Lambda^k_n$ with $\varepsilon_k \in \mathbb{F} \setminus \{0\}$ and $k \in \{0, 1, \ldots, n\}$, which is an identity mapping in projective geometry. For sure, in the case of a non-degenerate pair of polarities $(\hat{\pi}, \hat{\rho})$, the product of the two transformations never vanishes,

$$\hat{\pi}\hat{\rho}(X) = \hat{\rho}\hat{\pi}(X) \neq \mathbf{0} \qquad \forall\, X \in \Lambda_n \setminus \{\mathbf{0}\}. \tag{7.165}$$

This is why we can choose equation (7.159) as the condition for two degenerate polarities to be naturally associated. In doing so, we extend the concept of what naturally associated, degenerate polarities are. All degenerate and naturally associated polarities according to Definition 7.15 are included in the extended concept. But beyond the degenerate pairs of Definition 7.15, there are now more pairs fulfilling the condition of equation (7.159). E. g., the vanishing polarity $\hat{\pi} \equiv \mathbf{0}$ or $\hat{\rho} \equiv \mathbf{0}$ is naturally associated to every degenerate but non-vanishing polarity $\hat{\rho}$ or $\hat{\pi}$ respectively.

Definition 7.21 (Pairs of Naturally Associated Polarities). Two generic non-null polarities $\hat{\pi}$ and $\hat{\rho}$ according to Definition 7.1 form *a pair of naturally associated polarities* $(\hat{\pi}, \hat{\rho})$, if and only if

(D1) in case one of the two polarities is non-degenerate, then both polarities $\hat{\pi}$ and $\hat{\rho}$ are non-degenerate and represent the same transformation according to Theorem 4.39; or

(D2) in case one of the two polarities is degenerate, then $\hat{\pi}$ and $\hat{\rho}$ both are degenerate and satisfy

$$\hat{\pi}\hat{\rho}(X) = \hat{\rho}\hat{\pi}(X) = \mathbf{0} \qquad \forall\, X \in \Lambda_n. \tag{7.166}$$

Clearly, the version of Definition 7.21 is more simple than the preliminary version of Definition 7.15. And it has an impact on how the degenerate pairs of naturally associated polarities are represented in harmonic orthonormal systems of bases. Theorem 7.19 is extended and simplified to

Theorem 7.22 (Harmonic Orthonormal System of Bases III). *Let* $(\hat{\pi}, \hat{\rho})$ *represent a pair of two degenerate and naturally associated polarities according to Definition 7.21 with* $\lambda, \mu \in \{1, -1\}$. *Then there exists a system of bases* $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$, *which forms an harmonic model of projective algebra* Λ_n *according to Definition 4.26 and which is at the same time orthonormal with respect to the quadratic forms* $Q^{1+}_{\hat{\pi}}$ *and* $Q^{1-}_{\hat{\rho}}$ *and the bilinear forms* $B^{1+}_{\hat{\pi}}$ *and* $B^{1-}_{\hat{\rho}}$ *with*

$$Q^{1+}_{\hat{\pi}}(P_{\mathbf{b}}) \cdot Q^{1-}_{\hat{\rho}}(E_{\mathbf{b}}) = \eta^{+}_{\mathbf{b}} \eta^{-}_{\mathbf{b}} = 0 \tag{7.167}$$

for all binary numbers $\mathbf{b}$ *with* $S(\mathbf{b}) = 1$. *The signatures* $S^{+}(s^{+}, t^{+}, u^{+})$ *and* $S^{-}(s^{-}, t^{-}, u^{-})$ *satisfy the conditions*

$$2n \geq u^{+} + u^{-} \geq n, \qquad u^{-} \geq s^{+} + t^{+}, \qquad u^{+} \geq s^{-} + t^{-}. \tag{7.168}$$

Proof. According to Definition and Theorem 7.12, there is an orthonormal basis $\{P^{(1)}_{\mathbf{b}}\} \subset \Lambda_n$ with respect to the quadratic form $Q^{1+}_{\hat{\pi}}$ and the bilinear form $B^{1+}_{\hat{\pi}}$ with signature $S^{+}(s^{+}, t^{+}, u^{+})$,

$$\begin{aligned} Q^{1+}_{\hat{\pi}}(P^{(1)}_{{}_l\mathbf{u}}) &= \hat{\beta}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^{+}_{{}_l\mathbf{u}} \qquad\qquad S(\mathbf{u}) = n \\ &= \begin{cases} 1, & 1 \leq l \leq s^{+}, \\ -1, & s^{+} + 1 \leq l \leq s^{+} + t^{+}, \\ 0, & s^{+} + t^{+} + 1 \leq l \leq s^{+} + t^{+} + u^{+} = n, \end{cases} \end{aligned} \tag{7.169}$$

and an orthonormal basis $\{E^{(2)}_{\mathbf{b}}\} \subset \Lambda_n$ with respect to the quadratic form $Q^{1-}_{\hat{\rho}}$ and the bilinear form $B^{1-}_{\hat{\rho}}$ with signature $S^{-}(s^{-}, t^{-}, u^{-})$

$$\begin{aligned} Q^{1-}_{\hat{\rho}}(E^{(2)}_{{}_l\mathbf{u}}) &= (-1)^{n-1} \hat{\gamma}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^{-}_{{}_l\mathbf{u}} \qquad\qquad S(\mathbf{u}) = n \\ &= \begin{cases} 0, & 1 \leq l \leq u^{-}, \\ -1, & u^{-} + 1 \leq l \leq u^{-} + t^{-}, \\ 1, & u^{-} + t^{-} + 1 \leq l \leq u^{-} + t^{-} + s^{-} = n. \end{cases} \end{aligned} \tag{7.170}$$

By precondition, $\hat{\pi}$ and $\hat{\rho}$ are degenerate, i.e. $1 \leq u^{+}, u^{-} \leq n$.

Let $\{P^{(1)}_{\mathbf{b}}\}, \{E^{(1)}_{\mathbf{b}}\} \subset \Lambda_n$ denote the system of bases forming an harmonic model of projective algebra Λ_n and let $\{P^{(2)}_{\mathbf{b}}\}, \{E^{(2)}_{\mathbf{b}}\} \subset \Lambda_n$ denote the system of bases forming a second harmonic model of the projective algebra Λ_n.

We divide the systems of bases into two times three sets each,

$$\mathcal{B}^{+}_{1s} := \left\{ P^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ 1 \leq l \leq s^{+} \right\}, \tag{7.171}$$

$$\mathcal{B}^{+}_{1t} := \left\{ P^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^{+} + 1 \leq l \leq s^{+} + t^{+} \right\}, \tag{7.172}$$

$$\mathcal{B}^{+}_{1u} := \left\{ P^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^{+} + t^{+} + 1 \leq l \leq s^{+} + t^{+} + u^{+} = n \right\}, \tag{7.173}$$

$$\mathcal{B}^{-}_{1(s)} := \left\{ E^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ 1 \leq l \leq s^{+} \right\}, \tag{7.174}$$

$$\mathcal{B}^-_{1(t)} := \left\{ E^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^+ + 1 \le l \le s^+ + t^+ \right\}, \tag{7.175}$$

$$\mathcal{B}^-_{1(u)} := \left\{ E^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^+ + t^+ + 1 \le l \le s^+ + t^+ + u^+ = n \right\}, \tag{7.176}$$

$$\mathcal{B}^-_{2u} := \left\{ E^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ 1 \le l \le u^- \right\}, \tag{7.177}$$

$$\mathcal{B}^-_{2t} := \left\{ E^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ u^- + 1 \le l \le u^- + t^- \right\}, \tag{7.178}$$

$$\mathcal{B}^-_{2s} := \left\{ E^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ u^- + t^- + 1 \le l \le u^- + t^- + s^- = n \right\}, \tag{7.179}$$

$$\mathcal{B}^+_{2(u)} := \left\{ P^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ 1 \le l \le u^- \right\}, \tag{7.180}$$

$$\mathcal{B}^+_{2(t)} := \left\{ P^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ u^- + 1 \le l \le u^- + t^- \right\}, \tag{7.181}$$

$$\mathcal{B}^+_{2(s)} := \left\{ P^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ u^- + t^- + 1 \le l \le u^- + t^- + s^- = n \right\}, \tag{7.182}$$

and get the following relations,

$$\begin{aligned} \Lambda^{1+}_n &= \mathrm{span}(\mathcal{B}^+_{1s}) \oplus \mathrm{span}(\mathcal{B}^+_{1t}) \oplus \mathrm{span}(\mathcal{B}^+_{1u}) \\ &= \mathrm{span}(\mathcal{B}^+_{2(u)}) \oplus \mathrm{span}(\mathcal{B}^+_{2(t)}) \oplus \mathrm{span}(\mathcal{B}^+_{2(s)}), \end{aligned} \tag{7.183}$$

$$\begin{aligned} \Lambda^{1-}_n &= \mathrm{span}(\mathcal{B}^-_{2u}) \oplus \mathrm{span}(\mathcal{B}^-_{2t}) \oplus \mathrm{span}(\mathcal{B}^-_{2s}) \\ &= \mathrm{span}(\mathcal{B}^-_{1(s)}) \oplus \mathrm{span}(\mathcal{B}^-_{1(t)}) \oplus \mathrm{span}(\mathcal{B}^-_{1(u)}), \end{aligned} \tag{7.184}$$

$$\mathrm{span}(\mathcal{B}^+_{1u}) \supset \mathrm{span}(\mathcal{B}^+_{2(t)}) \oplus \mathrm{span}(\mathcal{B}^+_{2(s)}), \tag{7.185}$$

$$\mathrm{span}(\mathcal{B}^-_{2u}) \supset \mathrm{span}(\mathcal{B}^-_{1(s)}) \oplus \mathrm{span}(\mathcal{B}^-_{1(t)}). \tag{7.186}$$

The relations of equations (7.183) and (7.184) are obvious. In order to prepare the proof of equation (7.185), we collect with $S(\mathbf{b}) = 1$ and $S(\mathbf{u}) = n$ the identities

$$\hat\pi(P^{(1)}_{\mathbf{b}}) = \sum_{S(\mathbf{c})=1} \hat\beta_{\mathbf{bc}} E^{(1)}_{\mathbf{c}} = \hat\beta_{\mathbf{bb}} E^{(1)}_{\mathbf{b}} = \eta^+_{\mathbf{b}} E^{(1)}_{\mathbf{b}}, \tag{7.187}$$

$$\hat\rho(E^{(2)}_{\mathbf{b}}) = \sum_{S(\mathbf{c})=1} \hat\gamma_{\mathbf{bc}} P^{(2)}_{\mathbf{c}} = \hat\gamma_{\mathbf{bb}} P^{(2)}_{\mathbf{b}} = (-1)^{n-1} \eta^-_{\mathbf{b}} P^{(2)}_{\mathbf{b}}, \tag{7.188}$$

$$X = \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} P^{(1)}_{{}_l\mathbf{u}} \in \mathrm{span}(\mathcal{B}^+_{1s}) \oplus \mathrm{span}(\mathcal{B}^+_{1t}), \tag{7.189}$$

$$\begin{aligned} \hat\pi(X) &= \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \cdot \hat\pi(P^{(1)}_{{}_l\mathbf{u}}) = \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \hat\beta_{{}_l\mathbf{u}\,{}_l\mathbf{u}} \cdot E^{(1)}_{{}_l\mathbf{u}} \\ &= \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \eta^+_{{}_l\mathbf{u}} \cdot E^{(1)}_{{}_l\mathbf{u}} \in \mathrm{span}(\mathcal{B}^-_{1(s)}) \oplus \mathrm{span}(\mathcal{B}^-_{1(t)}) \end{aligned} \tag{7.190}$$

and the basis transformation

$$E^{(1)}_{{}_l\mathbf{u}} = \sum_{m=1}^{n} \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} \cdot E^{(2)}_{{}_m\mathbf{u}}, \tag{7.191}$$

which is an even automorphism ρ in Λ_n^{1-}. By precondition $(\hat{\pi}, \hat{\rho})$ represents a pair of two degenerate and naturally associated polarities. According to Definition 7.21 we have for all X of equation (7.189)

$$\begin{aligned}
\mathbf{0} = \hat{\rho}\hat{\pi}(X) &= \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \eta^+_{{}_l\mathbf{u}} \cdot \hat{\rho}(E^{(1)}_{{}_l\mathbf{u}}) \\
&= \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \eta^+_{{}_l\mathbf{u}} \sum_{m=1}^{n} \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} \cdot \hat{\rho}(E^{(2)}_{{}_m\mathbf{u}}) \\
&= \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \eta^+_{{}_l\mathbf{u}} \sum_{m=1}^{n} \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} (-1)^{n-1} \eta^-_{{}_m\mathbf{u}} P^{(2)}_{{}_m\mathbf{u}} \\
&= (-1)^{n-1} \sum_{m=u^-+1}^{n} \left[\sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \eta^+_{{}_l\mathbf{u}} \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} \eta^-_{{}_m\mathbf{u}} \right] P^{(2)}_{{}_m\mathbf{u}}
\end{aligned} \tag{7.192}$$

$$\Longleftrightarrow$$

$$0 = \sum_{l=1}^{s^+ + t^+} \mu_{{}_l\mathbf{u}} \eta^+_{{}_l\mathbf{u}} \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} \eta^-_{{}_m\mathbf{u}} \qquad \forall\, X \in \operatorname{span}(\mathcal{B}^+_{1s}) \oplus \operatorname{span}(\mathcal{B}^+_{1t}) \tag{7.193}$$

$$\Longleftrightarrow$$

$$0 = \eta^+_{{}_l\mathbf{u}} \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} \eta^-_{{}_m\mathbf{u}} \qquad \forall\, l, m \text{ with} \begin{cases} 1 \le l \le s^+ + t^+ \\ u^- + 1 \le m \le n \end{cases} \tag{7.194}$$

$$\Longleftrightarrow$$

$$0 = \gamma_{{}_l\mathbf{u}\,{}_m\mathbf{u}} \qquad \forall\, l, m \text{ with} \begin{cases} 1 \le l \le s^+ + t^+ \\ u^- + 1 \le m \le n \end{cases} \tag{7.195}$$

$$\Longleftrightarrow$$

$$\hat{\pi}(X) \in \operatorname{span}(\mathcal{B}^-_{2u}) \qquad \forall\, X \in \operatorname{span}(\mathcal{B}^+_{1s}) \oplus \operatorname{span}(\mathcal{B}^+_{1t}) \tag{7.196}$$

$$\Longleftrightarrow$$

$$\operatorname{span}(\mathcal{B}^+_{1u}) \supset \operatorname{span}(\mathcal{B}^+_{2(t)}) \oplus \operatorname{span}(\mathcal{B}^+_{2(s)}). \tag{7.197}$$

The relation of equation (7.186) is proven along equivalent considerations.

From equations (7.185) and (7.186) we get

$$u^- \ge s^+ + t^+ = n - u^+, \tag{7.198}$$

$$u^+ \ge s^- + t^- = n - u^-, \tag{7.199}$$

and thus

$$2n \ge u^+ + u^- \ge n. \tag{7.200}$$

The last three equations correspond to the equations (7.168) of Theorem 7.22.

We are now in the position to define an harmonic orthonormal system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$ for the degenerate and naturally associated pair of polarities $(\hat{\pi}, \hat{\rho})$. From equation (7.199) we get

$$h := u^+ - (s^- + t^-) \geq 0. \tag{7.201}$$

For the h-dimensional subspace

$$N^+ := \operatorname{span}(\mathcal{B}^+_{1u}) \setminus \operatorname{span}(\mathcal{B}^+_{2(t)} \cup \mathcal{B}^+_{2(s)}) \tag{7.202}$$

choose any basis

$$\mathcal{B}^+_{u(h)} := \left\{ P_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^+ + t^+ + 1 \leq l \leq s^+ + t^+ + h \right\}. \tag{7.203}$$

In case of $h = 0$, the subspace $\mathcal{B}^+_{u(h)}$ is the empty set $\{\}$. For the $(s^- + t^-)$-dimensional subspace

$$\operatorname{span}(\mathcal{B}^+_{2(t)} \cup \mathcal{B}^+_{2(s)}) \tag{7.204}$$

we use the basis 1-vectors

$$\mathcal{B}^+_{u(s^- + t^-)} := \left\{ P_{{}_l\mathbf{u}} := P^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^+ + t^+ + h + 1 \leq l \leq n \right\}. \tag{7.205}$$

By precondition, the quadratic form $\mathcal{Q}^{1+}_{\hat{\pi}}$ of equation (7.169) is vanishing on the set of basis 1-vectors $\mathcal{B}^+_{1u}$. The quadratic form $\mathcal{Q}^{1+}_{\hat{\pi}}$ then vanishes on any basis of $\operatorname{span}(\mathcal{B}^+_{1u})$. This is why it also vanishes on $\mathcal{B}^+_{u(h)}$, on $\mathcal{B}^+_{u(s^- + t^-)}$ and on $\mathcal{B}^+_{u(h)} \cup \mathcal{B}^+_{u(s^- + t^-)}$.

The basis of 1-vectors in the plus approach,

$$\mathcal{B}^+_s := \left\{ P_{{}_l\mathbf{u}} := P^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ 1 \leq l \leq s^+ \right\}, \tag{7.206}$$

$$\mathcal{B}^+_t := \left\{ P_{{}_l\mathbf{u}} := P^{(1)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ s^+ + 1 \leq l \leq s^+ + t^+ \right\}, \tag{7.207}$$

$$\mathcal{B}^+_u := \mathcal{B}^+_{u(h)} \cup \mathcal{B}^+_{u(s^- + t^-)}, \tag{7.208}$$

$$\{P_\mathbf{b} \,|\, S(\mathbf{b}) = 1\} = \mathcal{B}^+_s \cup \mathcal{B}^+_t \cup \mathcal{B}^+_u \tag{7.209}$$

and its harmonic complement $\{E_\mathbf{b} \,|\, S(\mathbf{b}) = 1\}$ according to Definition 4.26 in the minus approach,

$$\mathcal{B}^-_u := \left\{ E_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ 1 \leq l \leq u^- \right\}, \tag{7.210}$$

$$E_{{}_l\mathbf{u}} = E^{(1)}_{{}_l\mathbf{u}}, \qquad 1 \leq l \leq s^+ + t^+, \tag{7.211}$$

$$\mathcal{B}^-_t := \left\{ E_{{}_l\mathbf{u}} = E^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ u^- + 1 \leq l \leq u^- + t^- \right\}, \tag{7.212}$$

$$\mathcal{B}^-_s := \left\{ E_{{}_l\mathbf{u}} = E^{(2)}_{{}_l\mathbf{u}} \,\middle|\, l \in \mathbb{N},\ u^- + t^- + 1 \leq l \leq n \right\}, \tag{7.213}$$

$$\{E_\mathbf{b} \,|\, S(\mathbf{b}) = 1\} = \mathcal{B}^-_u \cup \mathcal{B}^-_t \cup \mathcal{B}^-_s \tag{7.214}$$

form the harmonic orthonormal system of bases for the degenerate and naturally associated pair of polarities $(\hat{\pi}, \hat{\rho})$. Their quadratic forms $Q^{1+}_{\hat{\pi}}$ with signature $S^+(s^+, t^+, u^+)$ and $Q^{1-}_{\hat{\rho}}$ with signature $S^-(s^-, t^-, u^-)$ are

$$Q^{1+}_{\hat{\pi}}(P_{{}_l\mathbf{u}}) = \hat{\beta}_{{}_l\mathbf{u}{}_l\mathbf{u}} = \eta^+_{{}_l\mathbf{u}} \qquad\qquad S(\mathbf{u}) = n \tag{7.215}$$

$$= \begin{cases} 1, & 1 \le l \le s^+, \\ -1, & s^+ + 1 \le l \le s^+ + t^+, \\ 0, & s^+ + t^+ + 1 \le l \le s^+ + t^+ + u^+ = n, \end{cases}$$

and

$$Q_{\hat{\rho}}^{1-}(E_{{}_l\mathbf{u}}) = (-1)^{n-1}\hat{\gamma}_{{}_l\mathbf{u}\,{}_l\mathbf{u}} = \eta^-_{{}_l\mathbf{u}} \qquad S(\mathbf{u}) = n \tag{7.216}$$

$$= \begin{cases} 0, & 1 \le l \le u^-, \\ -1, & u^- + 1 \le l \le u^- + t^-, \\ 1, & u^- + t^- + 1 \le l \le u^- + t^- + s^- = n. \end{cases}$$

With equations (7.185) and (7.186) and the quadratic forms (7.215) and (7.216) we have equation (7.167) of Theorem 7.22. □

Corollary 7.23. *For a naturally associated pair of polarities* $(\hat{\pi}, \hat{\rho})$ *according to Definition 7.21 with*

- $\lambda, \mu \in \{1, -1\}$,
- *an harmonic orthonormal system of bases* $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$,
- *signature* $S^+(s^+, t^+, u^+)$ *and signature* $S^-(s^-, t^-, u^-)$ *respectively*

we have:

$(\hat{\pi}, \hat{\rho})$ **non-degenerate.**

$$Q_{\hat{\pi}}^{1+}(P_{\mathbf{b}}) \cdot Q_{\hat{\rho}}^{1-}(E_{\mathbf{b}}) = \eta^+_{\mathbf{b}}\eta^-_{\mathbf{b}} = \frac{1}{\lambda\mu}, \qquad u^+ = u^- = 0, \tag{7.217}$$

$$s^+ + t^+ = n, \qquad s^- + t^- = n, \tag{7.218}$$

$$\det \hat{\pi} = \frac{1}{\mu\,\lambda^{n-1}}, \qquad \det \hat{\rho} = \frac{1}{\lambda\,\mu^{n-1}}. \tag{7.219}$$

$(\hat{\pi}, \hat{\rho})$ **degenerate.**

$$Q_{\hat{\pi}}^{1+}(P_{\mathbf{b}}) \cdot Q_{\hat{\rho}}^{1-}(E_{\mathbf{b}}) = \eta^+_{\mathbf{b}}\eta^-_{\mathbf{b}} = 0, \qquad n \le u^+ + u^- \le 2n, \tag{7.220}$$

$$s^+ + t^+ \le u^-, \qquad s^- + t^- \le u^+, \tag{7.221}$$

$$\det \hat{\pi} = 0, \qquad \det \hat{\rho} = 0. \tag{7.222}$$

Proof. Compare Definition and Theorem 7.18 as well as Theorem 7.22. □

7.8. Quadrics in $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$

In the following tables we will provide an overview on the different types of quadrics related to naturally associated pairs of polarities for the projective geometries $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$. The basic elements and the k-primitive geometric forms of grade m of these projective geometries were listed in Tables 4, 7 and 8. The quadrics are represented in the Tables 10 to 15 with respect to an harmonic orthonormal system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Lambda_n$, in each first line with respect to 1-vectors in the plus approach

$$X_{\overline{1}} = \sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}} P_{\mathbf{b}} \tag{7.223}$$

as

$$Q^{1+}_{\hat{\pi}}(X^{-}_{\bar{1}}) = \sum_{S(\mathbf{b})=1} \mu^{2}_{\mathbf{b}}\, \eta^{+}_{\mathbf{b}}, \tag{7.224}$$

in each second line with respect to 1-vectors in the minus approach

$$X^{-}_{\bar{1}} = \sum_{S(\mathbf{b})=1} \nu_{\mathbf{b}} E_{\mathbf{b}} \tag{7.225}$$

as

$$Q^{1-}_{\hat{\rho}}(X^{-}_{\bar{1}}) = \sum_{S(\mathbf{b})=1} \nu^{2}_{\mathbf{b}}\, \eta^{-}_{\mathbf{b}}. \tag{7.226}$$

Compare also Definition and Theorem 7.5.

Table 10 displays all different types of quadrics from point-plane projective geometry $\mathcal{P}_2$ in *one* line (L) and all different types of quadrics from line-line projective geometry $\mathcal{P}_2$ in *one* incident point-plane pair (PPP). The non-degenerate quadrics in block (1) and (2) belong to a non-degenerate pair $(\hat{\pi}, \hat{\rho})$ of naturally associated polarities. The degenerate quadrics in block (3) to (5) belong to a degenerate pair $(\hat{\pi}, \hat{\rho})$ of naturally associated polarities. Block (3) displays the quadric belonging to the cases $u^{+} + u^{-} = 2$, for the blocks (4) and (5) we have $u^{+} + u^{-} > 2$. Not listed is the trivial case

$$S^{+}(0,0,2), \qquad\qquad S^{-}(0,0,2). \tag{7.227}$$

The parameters λ and μ were introduced in the definitions of the algebra homomorphisms. Compare equation (4.50) of Definition and Theorem 4.14, equation (4.76) of Definition and Theorem 4.15, equation (4.54) of Definition and Theorem 4.14 and equation (4.80) of Definition and Theorem 4.15. Block (2) of Table 10 gives an example, that we indeed need the parameters λ and μ, in order to describe all the different quadrics in both approaches simultaneously. This need is, of course, already obvious through Definition and Theorem 7.18 and through Theorem 4.39. The parameters λ and μ are not necessary to describe all the different quadrics in just *one* approach, but as soon as we want to describe them from *both* approaches, i. e. in space and counterspace simultaneously, we have to work with them.

Another example for the necessity to define the algebra homomorphisms with the parameters λ and μ can be found in point-plane projective geometry $\mathcal{P}_4$ of space. See Table 14, block (3).

Table 11, 12 and 13 display the non-degenerate and degenerate conic sections from point-line projective geometry $\mathcal{P}_3$ in the planar field and the non-degenerate and degenerate cones from plane-line projective geometry $\mathcal{P}_3$ in the centric bundle. The dimension $n = 3$ of these geometries is odd, i. e., the planar field or the centric bundle is one-sided and cannot distinguish between two different orientations. But instead, each type of conic section or cone can be displayed in all four combinations of $\lambda, \mu \in \{1, -1\}$. The remark 'incident' in brackets in the first two lines of each box means, the conic sections or the cones in space and the conic sections or cones in counterspace coincide respectively.

	Signatures $\mathcal{S}^{\pm}$	λ, μ, $\det\hat{\pi}$, $\det\hat{\rho}$	Quadric
(1)	**L: 1 pair of complex conjugated points & 1 pair of complex conjugated planes (incident)**		
	PPP: 1 pair of complex conj. lines in the plus approach & 1 pair complex conjug. lines in the minus approach (incident)		
	$S^{+}(2,0,0)$	$\lambda=\mu=\pm 1$	$\mu_{01}^2+\mu_{10}^2=0$
	$S^{-}(2,0,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$\nu_{01}^2+\nu_{10}^2=0$
	$S^{+}(0,2,0)$	$\lambda=\mu=\pm 1$	$-\mu_{01}^2-\mu_{10}^2=0$
	$S^{-}(0,2,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$-\nu_{01}^2-\nu_{10}^2=0$
(2)	**L: 1 pair of real points & 1 pair of real planes (incident)**		
	PPP: 1 pair of real lines in the plus approach & 1 pair of real lines in the minus approach (incident)		
	$S^{+}(1,1,0)$	$\lambda=-\mu=\pm 1$	$\mu_{01}^2-\mu_{10}^2=0$
	$S^{-}(1,1,0)$	$\det\hat{\pi}=\det\hat{\rho}=-1$	$-\nu_{01}^2+\nu_{10}^2=0$
(3)	**L: 1 double point & 1 double plane (incident)**		
	PPP: 1 double line in the plus approach & 1 double line in the minus approach (incident)		
	$S^{+}(1,0,1)$		$\mu_{01}^2=0$
	$S^{-}(0,1,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{10}^2=0$
	$S^{+}(0,1,1)$		$-\mu_{10}^2=0$
	$S^{-}(1,0,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{01}^2=0$
(4)	**L: 1 double point & the line as pencil of planes (incident)**		
	PPP: 1 double line in the plus approach & the point-plane-pair in the minus approach (incident)		
	$S^{+}(1,0,1)$		$\mu_{01}^2=0$
	$S^{-}(0,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
	$S^{+}(0,1,1)$		$-\mu_{10}^2=0$
	$S^{-}(0,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
(5)	**L: The line as range of points & 1 double plane (incident)**		
	PPP: The point-plane-pair in the plus approach & 1 double line in the minus approach (incident)		
	$S^{+}(0,0,2)$		
	$S^{-}(1,0,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{01}^2=0$
	$S^{+}(0,0,2)$		
	$S^{-}(0,1,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{10}^2=0$

Table 10. Quadrics in the line $\mathcal{P}_2$ (L) or in the incident point-plane-pair $\mathcal{P}_2$ (PPP)

	Signatures $\mathcal{S}^{\pm}$	λ, μ, $\det\hat{\pi}$, $\det\hat{\rho}$	Quadric
(1)			**PF: anisotropic conic section (incident)** **CB: anisotropic cone (incident)**
	$S^{+}(3,0,0)$	$\lambda=\mu=1$	$\mu_{001}^2+\mu_{010}^2+\mu_{100}^2=0$
	$S^{-}(3,0,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$\nu_{001}^2+\nu_{010}^2+\nu_{100}^2=0$
	$S^{+}(0,3,0)$	$\lambda=\mu=-1$	$-\mu_{001}^2-\mu_{010}^2-\mu_{100}^2=0$
	$S^{-}(0,3,0)$	$\det\hat{\pi}=\det\hat{\rho}=-1$	$-\nu_{001}^2-\nu_{010}^2-\nu_{100}^2=0$
	$S^{+}(0,3,0)$	$\lambda=-\mu=1$	$-\mu_{001}^2-\mu_{010}^2-\mu_{100}^2=0$
	$S^{-}(3,0,0)$	$\det\hat{\pi}=-\det\hat{\rho}=-1$	$\nu_{001}^2+\nu_{010}^2+\nu_{100}^2=0$
	$S^{+}(3,0,0)$	$\lambda=-\mu=-1$	$\mu_{001}^2+\mu_{010}^2+\mu_{100}^2=0$
	$S^{-}(0,3,0)$	$\det\hat{\pi}=-\det\hat{\rho}=1$	$-\nu_{001}^2-\nu_{010}^2-\nu_{100}^2=0$
(2)			**PF: oval conic section (incident)** **CB: oval cone (incident)**
	$S^{+}(1,2,0)$	$\lambda=\mu=1$	$\mu_{001}^2-\mu_{010}^2-\mu_{100}^2=0$
	$S^{-}(1,2,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$\nu_{001}^2-\nu_{010}^2-\nu_{100}^2=0$
	$S^{+}(2,1,0)$	$\lambda=\mu=-1$	$\mu_{001}^2+\mu_{010}^2-\mu_{100}^2=0$
	$S^{-}(2,1,0)$	$\det\hat{\pi}=\det\hat{\rho}=-1$	$\nu_{001}^2+\nu_{010}^2-\nu_{100}^2=0$
	$S^{+}(2,1,0)$	$\lambda=-\mu=1$	$\mu_{001}^2+\mu_{010}^2-\mu_{100}^2=0$
	$S^{-}(1,2,0)$	$\det\hat{\pi}=-\det\hat{\rho}=-1$	$-\nu_{001}^2-\nu_{010}^2+\nu_{100}^2=0$
	$S^{+}(1,2,0)$	$\lambda=-\mu=-1$	$\mu_{001}^2-\mu_{010}^2-\mu_{100}^2=0$
	$S^{-}(2,1,0)$	$\det\hat{\pi}=-\det\hat{\rho}=1$	$-\nu_{001}^2+\nu_{010}^2+\nu_{100}^2=0$
(3)			**PF: 1 pair of complex conjugated points & 1 double line (incident)** **CB: 1 pair of complex conjugated lines & 1 double plane (incident)**
	$S^{+}(2,0,1)$		$\mu_{001}^2+\mu_{010}^2=0$
	$S^{-}(1,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{100}^2=0$
	$S^{+}(2,0,1)$		$\mu_{001}^2+\mu_{010}^2=0$
	$S^{-}(0,1,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{100}^2=0$
	$S^{+}(0,2,1)$		$-\mu_{001}^2-\mu_{010}^2=0$
	$S^{-}(1,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{100}^2=0$
	$S^{+}(0,2,1)$		$-\mu_{001}^2-\mu_{010}^2=0$
	$S^{-}(0,1,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{100}^2=0$

TABLE 11. Non-degenerate and degenerate conic sections in the planar field $\mathcal{P}_3$ (PF) as well as non-degenerate and degenerate cones in the centric bundle $\mathcal{P}_3$ (CB)

	Signatures $\mathcal{S}^{\pm}$	λ, μ, $\det \hat{\pi}$, $\det \hat{\rho}$	Quadric
(4)			**PF: 1 pair of real points & 1 double line (incident)**
			CB: 1 pair of real lines & 1 double plane (incident)
	$S^{+}(1,1,1)$		$\mu_{001}^2 - \mu_{010}^2 = 0$
	$S^{-}(1,0,2)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{100}^2 = 0$
	$S^{+}(1,1,1)$		$\mu_{001}^2 - \mu_{010}^2 = 0$
	$S^{-}(0,1,2)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$-\nu_{100}^2 = 0$
(5)			**PF: 1 double point & 1 pair of complex conj. lines (incident)**
			CB: 1 double line & 1 pair of complex conj. planes (incident)
	$S^{+}(1,0,2)$		$\mu_{100}^2 = 0$
	$S^{-}(2,0,1)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{001}^2 + \nu_{010}^2 = 0$
	$S^{+}(0,1,2)$		$-\mu_{100}^2 = 0$
	$S^{-}(2,0,1)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{001}^2 + \nu_{010}^2 = 0$
	$S^{+}(1,0,2)$		$\mu_{100}^2 = 0$
	$S^{-}(0,2,1)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$-\nu_{001}^2 - \nu_{010}^2 = 0$
	$S^{+}(0,1,2)$		$-\mu_{100}^2 = 0$
	$S^{-}(0,2,1)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$-\nu_{001}^2 - \nu_{010}^2 = 0$
(6)			**PF: 1 double point & 1 pair of real lines (incident)**
			CB: 1 double line & 1 pair of real planes (incident)
	$S^{+}(1,0,2)$		$\mu_{100}^2 = 0$
	$S^{-}(1,1,1)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{001}^2 - \nu_{010}^2 = 0$
	$S^{+}(0,1,2)$		$-\mu_{100}^2 = 0$
	$S^{-}(1,1,1)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{001}^2 - \nu_{010}^2 = 0$
(7)			**PF: 1 double point & 1 double line (incident)**
			CB: 1 double line & 1 double plane (incident)
	$S^{+}(1,0,2)$		$\mu_{100}^2 = 0$
	$S^{-}(1,0,2)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{001}^2 = 0$
	$S^{+}(1,0,2)$		$\mu_{100}^2 = 0$
	$S^{-}(0,1,2)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$-\nu_{001}^2 = 0$
	$S^{+}(0,1,2)$		$-\mu_{100}^2 = 0$
	$S^{-}(1,0,2)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$\nu_{001}^2 = 0$
	$S^{+}(0,1,2)$		$-\mu_{100}^2 = 0$
	$S^{-}(0,1,2)$	$\det \hat{\pi} = \det \hat{\rho} = 0$	$-\nu_{001}^2 = 0$

TABLE 12. Continuation from Table 11): Degenerate quadrics (conic sections) in the planar field $\mathcal{P}_3$ (PF) and degenerate quadrics (cones) in the centric bundle $\mathcal{P}_3$ (CB)

	Signatures $\mathcal{S}^{\pm}$	λ, μ, $\det\hat{\pi}$, $\det\hat{\rho}$	Quadric
(8)			**PF: 1 pair of complex conjugated points & the planar field of lines (incident)**
			CB: 1 pair of complex conjugated lines & the centric bundle of planes (incident)
	$S^{+}(2,0,1)$		$\mu_{001}^{2}+\mu_{010}^{2}=0$
	$S^{-}(0,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
	$S^{+}(0,2,1)$		$-\mu_{001}^{2}-\mu_{010}^{2}=0$
	$S^{-}(0,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
(9)			**PF: 1 pair of real points & the planar field of lines (incident)**
			CB: 1 pair of real lines & the centric bundle of planes (incident)
	$S^{+}(1,1,1)$		$\mu_{001}^{2}-\mu_{010}^{2}=0$
	$S^{-}(0,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
(10)			**PF: The planar field of points & 1 pair of complex conjugated lines (incident)**
			CB: The centric bundle of lines & 1 pair of complex conjugated planes (incident)
	$S^{+}(0,0,3)$		
	$S^{-}(2,0,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{001}^{2}+\nu_{010}^{2}=0$
	$S^{+}(0,0,3)$		
	$S^{-}(0,2,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{001}^{2}-\nu_{010}^{2}=0$
(11)			**PF: The planar field of points & 1 pair of real lines (incident)**
			CB: The centric bundle of lines & 1 pair of real planes (incident)
	$S^{+}(0,0,3)$		
	$S^{-}(1,1,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{001}^{2}-\nu_{010}^{2}=0$
(12)			**PF: 1 double point & the planar field of lines (incident)**
			CB: 1 double line & the centric bundle of planes (incident)
	$S^{+}(1,0,2)$		$\mu_{100}^{2}=0$
	$S^{-}(0,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
	$S^{+}(0,1,2)$		$-\mu_{100}^{2}=0$
	$S^{-}(0,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	
(13)			**PF: The planar field of points & 1 double line (incident)**
			CB: The centric bundle of lines & 1 double plane (incident)
	$S^{+}(0,0,3)$		
	$S^{-}(1,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{100}^{2}=0$
	$S^{+}(0,0,3)$		
	$S^{-}(0,1,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{100}^{2}=0$

TABLE 13. Continuation from Table 12): Degenerate quadrics (conic sections) in the planar field $\mathcal{P}_3$ (PL) and degenerate quadricsw (cones) in the centric bundle $\mathcal{P}_3$ (CB)

	Signatures $\mathcal{S}^{\pm}$	λ, μ, $\det\hat{\pi}$, $\det\hat{\rho}$	Quadric
(1)			**anisotropic quadric (incident)**
	$S^{+}(4,0,0)$	$\lambda=\mu=\pm1$	$\mu_{0001}^2+\mu_{0010}^2+\mu_{0100}^2+\mu_{1000}^2=0$
	$S^{-}(4,0,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$\nu_{0001}^2+\nu_{0010}^2+\nu_{0100}^2+\nu_{1000}^2=0$
	$S^{+}(0,4,0)$	$\lambda=\mu=\pm1$	$-\mu_{0001}^2-\mu_{0010}^2-\mu_{0100}^2-\mu_{1000}^2=0$
	$S^{-}(0,4,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$-\nu_{0001}^2-\nu_{0010}^2-\nu_{0100}^2-\nu_{1000}^2=0$
(2)			**ruled quadric (incident)**
	$S^{+}(2,2,0)$	$\lambda=\mu=\pm1$	$\mu_{0001}^2+\mu_{0010}^2-\mu_{0100}^2-\mu_{1000}^2=0$
	$S^{-}(2,2,0)$	$\det\hat{\pi}=\det\hat{\rho}=1$	$\nu_{0001}^2+\nu_{0010}^2-\nu_{0100}^2-\nu_{1000}^2=0$
(3)			**oval quadric (incident)**
	$S^{+}(3,1,0)$	$\lambda=-\mu=\pm1$	$\mu_{0001}^2+\mu_{0010}^2+\mu_{0100}^2-\mu_{1000}^2=0$
	$S^{-}(1,3,0)$	$\det\hat{\pi}=\det\hat{\rho}=-1$	$-\nu_{0001}^2-\nu_{0010}^2-\nu_{0100}^2+\nu_{1000}^2=0$
	$S^{+}(1,3,0)$	$\lambda=-\mu=\pm1$	$\mu_{0001}^2-\mu_{0010}^2-\mu_{0100}^2-\mu_{1000}^2=0$
	$S^{-}(3,1,0)$	$\det\hat{\pi}=\det\hat{\rho}=-1$	$-\nu_{0001}^2+\nu_{0010}^2+\nu_{0100}^2+\nu_{1000}^2=0$
(4)			**1 anisotropic conic section & 1 double plane (incident)**
	$S^{+}(3,0,1)$		$\mu_{0001}^2+\mu_{0010}^2+\mu_{0100}^2=0$
	$S^{-}(1,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{1000}^2=0$
	$S^{+}(3,0,1)$		$\mu_{0001}^2+\mu_{0010}^2+\mu_{0100}^2=0$
	$S^{-}(0,1,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{1000}^2=0$
	$S^{+}(0,3,1)$		$-\mu_{0001}^2-\mu_{0010}^2-\mu_{0100}^2=0$
	$S^{-}(1,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{1000}^2=0$
	$S^{+}(0,3,1)$		$-\mu_{0001}^2-\mu_{0010}^2-\mu_{0100}^2=0$
	$S^{-}(0,1,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{1000}^2=0$
(5)			**1 oval conic section & 1 double plane (incident)**
	$S^{+}(2,1,1)$		$\mu_{0001}^2+\mu_{0010}^2-\mu_{0100}^2=0$
	$S^{-}(1,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{1000}^2=0$
	$S^{+}(2,1,1)$		$\mu_{0001}^2+\mu_{0010}^2-\mu_{0100}^2=0$
	$S^{-}(0,1,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{1000}^2=0$
	$S^{+}(1,2,1)$		$\mu_{0001}^2-\mu_{0010}^2-\mu_{0100}^2=0$
	$S^{-}(1,0,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{1000}^2=0$
	$S^{+}(1,2,1)$		$\mu_{0001}^2-\mu_{0010}^2-\mu_{0100}^2=0$
	$S^{-}(0,1,3)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{1000}^2=0$
(6)			**1 double point & 1 anisotropic cone (incident)**
	$S^{+}(1,0,3)$		$\mu_{1000}^2=0$
	$S^{-}(3,0,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0001}^2+\nu_{0010}^2+\nu_{0100}^2=0$
	$S^{+}(0,1,3)$		$-\mu_{1000}^2=0$
	$S^{-}(3,0,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0001}^2+\nu_{0010}^2+\nu_{0100}^2=0$
	$S^{+}(1,0,3)$		$\mu_{1000}^2=0$
	$S^{-}(0,3,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0001}^2-\nu_{0010}^2-\nu_{0100}^2=0$
	$S^{+}(0,1,3)$		$-\mu_{1000}^2=0$
	$S^{-}(0,3,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0001}^2-\nu_{0010}^2-\nu_{0100}^2=0$

TABLE 14. Quadrics in projective space $\mathcal{P}_4$

	Signatures $\mathcal{S}^{\pm}$	λ, μ, $\det\hat{\pi}$, $\det\hat{\rho}$	Quadric
(7)			**1 double point & 1 oval cone (incident)**
	$S^{+}(1,0,3)$		$\mu_{1000}^2=0$
	$S^{-}(2,1,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0001}^2+\nu_{0010}^2-\nu_{0100}^2=0$
	$S^{+}(0,1,3)$		$-\mu_{1000}^2=0$
	$S^{-}(2,1,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0001}^2+\nu_{0010}^2-\nu_{0100}^2=0$
	$S^{+}(1,0,3)$		$\mu_{1000}^2=0$
	$S^{-}(1,2,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0001}^2-\nu_{0010}^2-\nu_{0100}^2=0$
	$S^{+}(0,1,3)$		$-\mu_{1000}^2=0$
	$S^{-}(1,2,1)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0001}^2-\nu_{0010}^2-\nu_{0100}^2=0$
(8)			**1 pair of complex conjugated points & 1 pair of complex conjugated planes (incident in one line)**
	$S^{+}(2,0,2)$		$\mu_{0001}^2+\mu_{0010}^2=0$
	$S^{-}(2,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0100}^2+\nu_{1000}^2=0$
	$S^{+}(2,0,2)$		$\mu_{0001}^2+\mu_{0010}^2=0$
	$S^{-}(0,2,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0100}^2-\nu_{1000}^2=0$
	$S^{+}(0,2,2)$		$-\mu_{0001}^2-\mu_{0010}^2=0$
	$S^{-}(2,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0100}^2+\nu_{1000}^2=0$
	$S^{+}(0,2,2)$		$-\mu_{0001}^2-\mu_{0010}^2=0$
	$S^{-}(0,2,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0100}^2-\nu_{1000}^2=0$
(9)			**1 pair of complex conjugated points & 1 pair of real planes (incident in one line)**
	$S^{+}(2,0,2)$		$\mu_{0001}^2+\mu_{0010}^2=0$
	$S^{-}(1,1,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0100}^2+\nu_{1000}^2=0$
	$S^{+}(0,2,2)$		$-\mu_{0001}^2-\mu_{0010}^2=0$
	$S^{-}(1,1,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0100}^2+\nu_{1000}^2=0$
(10)			**1 pair of real points & 1 pair of complex conjugated planes (incident in one line)**
	$S^{+}(1,1,2)$		$\mu_{0001}^2-\mu_{0010}^2=0$
	$S^{-}(2,0,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$\nu_{0100}^2+\nu_{1000}^2=0$
	$S^{+}(1,1,2)$		$\mu_{0001}^2-\mu_{0010}^2=0$
	$S^{-}(0,2,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0100}^2-\nu_{1000}^2=0$
(11)			**1 pair of real points & 1 pair of real planes (incident in one line)**
	$S^{+}(1,1,2)$		$\mu_{0001}^2-\mu_{0010}^2=0$
	$S^{-}(1,1,2)$	$\det\hat{\pi}=\det\hat{\rho}=0$	$-\nu_{0100}^2+\nu_{1000}^2=0$

Table 15. Continuation from Table 14: Quadrics in projective space $\mathcal{P}_4$

Block (1) and (2) display the non-degenerate quadrics in $\mathcal{P}_3$, the blocks (3) to (6) degenerate quadrics in $\mathcal{P}_3$ with $u^+ + u^- = 3$, blocks (7) to (11) degenerate quadrics in $\mathcal{P}_3$ with $u^+ + u^- = 4$ and the blocks (12) to (13) degenerate quadrics in $\mathcal{P}_3$ with $u^+ + u^- = 5$. Not listed is the trivial case

$$S^+(0,0,3), \qquad S^-(0,0,3). \tag{7.228}$$

Table 14 and 15 display the non-degenerate quadrics and the degenerate quadrics from point-plane projective geometry $\mathcal{P}_4$ in space. From the degenerate quadrics only those are listed satisfying the condition $u^+ + u^- = 4$. Not listed are the degenerate quadrics with $4 < u^+ + u^- \leq 8$, i. e.

$$u^+ + u^- = 5: \qquad S^+(3,0,1), \qquad S^-(0,0,4), \tag{7.229}$$

$$S^+(0,3,1), \qquad S^-(0,0,4), \tag{7.230}$$

$$S^+(2,1,1), \qquad S^-(0,0,4), \tag{7.231}$$

$$S^+(1,2,1), \qquad S^-(0,0,4), \tag{7.232}$$

$$S^+(0,0,4), \qquad S^-(3,0,1), \tag{7.233}$$

$$S^+(0,0,4), \qquad S^-(0,3,1), \tag{7.234}$$

$$S^+(0,0,4), \qquad S^-(2,1,1), \tag{7.235}$$

$$S^+(0,0,4), \qquad S^-(1,2,1), \tag{7.236}$$

$$S^+(2,0,2), \qquad S^-(1,0,3), \tag{7.237}$$

$$S^+(0,2,2), \qquad S^-(1,0,3), \tag{7.238}$$

$$S^+(1,1,2), \qquad S^-(1,0,3), \tag{7.239}$$

$$S^+(1,0,3), \qquad S^-(2,0,2), \tag{7.240}$$

$$S^+(1,0,3), \qquad S^-(0,2,2), \tag{7.241}$$

$$S^+(1,0,3), \qquad S^-(1,1,2), \tag{7.242}$$

$$S^+(2,0,2), \qquad S^-(0,1,3), \tag{7.243}$$

$$S^+(0,2,2), \qquad S^-(0,1,3), \tag{7.244}$$

$$S^+(1,1,2), \qquad S^-(0,1,3), \tag{7.245}$$

$$S^+(0,1,3), \qquad S^-(2,0,2), \tag{7.246}$$

$$S^+(0,1,3), \qquad S^-(0,2,2), \tag{7.247}$$

$$S^+(0,1,3), \qquad S^-(1,1,2); \tag{7.248}$$

$$u^+ + u^- = 6: \qquad S^+(2,0,2), \qquad S^-(0,0,4), \tag{7.249}$$

$$S^+(0,2,2), \qquad S^-(0,0,4), \tag{7.250}$$

$$S^+(1,1,2), \qquad S^-(0,0,4), \tag{7.251}$$

$$S^+(0,0,4), \qquad S^-(2,0,2), \tag{7.252}$$

$$S^+(0,0,4), \qquad S^-(0,2,2), \tag{7.253}$$

$$S^+(0,0,4), \qquad S^-(1,1,2), \tag{7.254}$$

$$S^+(1,0,3), \qquad S^-(1,0,3), \tag{7.255}$$

$$S^+(0,1,3), \qquad S^-(1,0,3), \tag{7.256}$$

$$S^+(1,0,3), \qquad S^-(0,1,3), \tag{7.257}$$

$$S^+(0,1,3), \qquad S^-(0,1,3); \tag{7.258}$$

$$u^+ + u^- = 7: \qquad S^+(1,0,3), \qquad S^-(0,0,4), \tag{7.259}$$

$$S^+(0,1,3), \qquad S^-(0,0,4), \tag{7.260}$$

$$S^+(0,0,4), \qquad S^-(1,0,3), \tag{7.261}$$

$$S^+(0,0,4), \qquad S^-(0,1,3); \tag{7.262}$$

$$u^+ + u^- = 8: \qquad S^+(0,0,4), \qquad S^-(0,0,4). \tag{7.263}$$

8. Transition to Clifford Double Algebras Γ_n and to Metric Geometries $\mathcal{M}_n$

ARTHUR CAYLEY in his paper *A sixth memoir upon the quantics* [Cay59] and FELIX KLEIN in the book *Vorlesungen über nicht-euklidische Geometrie* [Kle28] developed the construction of metric Cayley-Klein geometries. In order to do so, one is starting within the framework of projective geometry $\mathcal{P}_n$, singling out one, eventually degenerate, quadric, which is then defining the metrics in the corresponding Cayley-Klein geometry.

It was shown by several authors, how to represent the different metric Cayley-Klein geometries within the framework of Clifford algebras. With respect to this topic, we especially refer to [Gun11a, GDK19, Hav21, KH13, Kla14] and build upon the results presented there. The textbook in German by GERHARD KOWOL on projective and Cayley-Klein geometries [Kow09] provides a very good introduction and overview.

Since the signature of Clifford algebra is related to the absolute quadric mentioned above, we will show in this closing section, how to find the transition from projective algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ and its quadrics to Clifford double algebra $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$.

To begin with, we redisplay the definition of a Clifford algebra $Cl_n(+,\cdot,\ )$ for its own sake following again the lecture notes of DOUGLAS LUNDHOLM and LARS SVENSSON. [LS16, pp. 6-7]

Definition 8.1 (Clifford Algebra Cl_n). Let V denote a n-dimensional vector space with quadratic form Q; let $\mathcal{T}(V)$ display the tensor algebra over V and $\mathcal{I}_Q(V)$ the two-sided ideal generated by all elements of the form $v \otimes v - Q(v)$. The *Clifford algebra* $Cl_n(V,Q)$ is defined by quoting out the ideal $\mathcal{I}_Q(V)$ from the tensor algebra $\mathcal{T}(V)$,

$$Cl_n(V,Q) := \mathcal{T}(V)/\mathcal{I}_Q(V). \tag{8.1}$$

Corollary 8.2. *The* Clifford product

$$\begin{array}{ccc} Cl_n \times Cl_n & \longrightarrow & Cl_n \\ (A,B) & \longmapsto & AB \end{array} \tag{8.2}$$

is inherited from the tensor product in $\mathcal{T}(V)$. We denote it by juxtaposition. The Clifford product is unital, bilinear and associative, in general not commutative. In addition there is the contraction rule for the vectors v of the vector space V,

$$v^2 = \pm Q(v) \qquad \forall\ v \in V. \tag{8.3}$$

Proof. Compare [LS16, pp. 6-7]. □

Let us fix here the notation for the two Clifford products, which are going to be defined later on.

Notation 8.3 (Major and Minor Clifford Product). The Clifford double algebra $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ will carry two times the imprint of a Clifford algebra. This is why there are also two in general different Clifford products,

$$\begin{array}{ccc} \Gamma_n \times \Gamma_n & \longrightarrow & \Gamma_n \\ (A,B) & \longmapsto & AB \end{array} \qquad \begin{array}{ccc} \Gamma_n \times \Gamma_n & \xrightarrow{*} & \Gamma_n \\ (A,B) & \longmapsto & A * B \end{array} \tag{8.4}$$

The products are called *major Clifford product* (no sign) and *minor Clifford product* ($*$).

Notation 8.4 (Multiple Clifford Product Signs). For multiple Clifford products we use the notation

$$\prod_{l=1}^{m} X_l := X_1 X_2 \cdots X_m, \qquad \mathop{\ast}\limits_{l=1}^{m} X_l := X_1 * X_2 * \cdots * X_m. \tag{8.5}$$

In order to find the transition from projective geometry $\mathcal{P}_n$ with respect to a, for the moment, non-degenerate quadric $\mathcal{Q}$ to metric Cayley-Klein geometry, let $\Lambda_n(+,\cdot,\wedge,\vee)$ represent a projective $\mathbb{F}$-algebra[16] and let $(\hat{\pi},\hat{\rho})$ represent a pair of, for the moment, non-degenerate naturally associated polarities[17] with $\lambda,\mu \in \{1,-1\}$. Following Corollary 8.2, we require the major Clifford product

$$\begin{array}{ccc} \Gamma_n \times \Gamma_n & \longrightarrow & \Gamma_n \\ (A,B) & \longmapsto & AB \end{array} \tag{8.6}$$

to be unital,

$$\mathbf{1}^+ := \mathbf{Z}^+ = \mathbf{I}^-, \tag{8.7}$$

$$X_{\overline{0}} M = M X_{\overline{0}} := \alpha M \qquad \forall\ X_{\overline{0}} = \alpha \mathbf{1}^+,\ \alpha \in \mathbb{F},\ \forall\ M \in \Lambda_n \tag{8.8}$$

distributive,

$$A(B+C) = AB + AC \qquad \forall\ A,B,C \in \Lambda_n, \tag{8.9}$$

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[16] Compare Definition 5.1 and Definition 4.2.

[17] Compare Definition 7.21.

$$(A+B)C = AC + BC \qquad \forall\ A, B, C \in \Lambda_n, \tag{8.10}$$

and associative,

$$A(BC) = (AB)C \qquad \forall\ A, B, C \in \Lambda_n. \tag{8.11}$$

In addition, the product of two generic 1-vectors decomposes by definition into the sum of a scalar and a 2-vector according to

$$\begin{aligned} AB &:= B^{1+}_{\hat{\pi}}(A,B)\mathbf{Z}^+ + A \wedge B \qquad \forall\ A, B \in \Lambda^{1+}_n, \\ &= B^{1+}_{\hat{\pi}}(A,B)\mathbf{1}^+ + A \wedge B \end{aligned} \tag{8.12}$$

which includes the contraction rule for generic 1-vectors,

$$\begin{aligned} A^2 = AA &= B^{1+}_{\hat{\pi}}(A,A)\mathbf{Z}^+ = Q^{1+}_{\hat{\pi}}(A)\mathbf{Z}^+ \qquad \forall\ A \in \Lambda^{1+}_n. \\ &= B^{1+}_{\hat{\pi}}(A,A)\mathbf{1}^+ = Q^{1+}_{\hat{\pi}}(A)\mathbf{1}^+ \end{aligned} \tag{8.13}$$

And last, for any orthogonal set $\mathcal{S} = \{X_1, X_2, \ldots, X_m\} \subset \Lambda^{1+}_n$ of m homogeneous vectors of grade 1 in the plus approach, we require the major Clifford and the major exterior products to be the same,

$$\prod_{l=1}^{m} X_l := \bigwedge_{l=1}^{m} X_l \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \mathcal{S}^+ \text{ is a orthogonal system with} \\ \text{respect to the bilinear form } B^{1+}_{\hat{\pi}}. \end{array} \right. \tag{8.14}$$

This is how the Clifford algebra $\Gamma^+_n(+,\cdot,\ )$ with the one-element $\mathbf{1}^+$ (defined in equations (8.7) and (8.8)) is constructed within the framework of projective geometry $\mathcal{P}_n$ and with respect to the non-degenerate quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$. In order to construct the dual Clifford algebra $\Gamma^-_n(+,\cdot,*)$ and its minor Clifford product within the same framework, we use the commutative diagram of mappings,

$$\begin{tikzcd}
\begin{array}{c} \Gamma^+_n \otimes \Gamma^+_n \\ (A,B) \end{array} \arrow[r, "\text{major Clifford product}"] \arrow[d, "\hat{\pi} \otimes \hat{\pi}"] & \begin{array}{c} \Gamma^+_n \\ AB \end{array} \arrow[d, "\hat{\pi}"] \\
\begin{array}{c} \Gamma^-_n \otimes \Gamma^-_n \\ (\hat{\pi}(A), \hat{\pi}(B)) \end{array} \arrow[r, "\text{minor Clifford product}"] & \begin{array}{c} \Gamma^-_n \\ \hat{\pi}(A) * \hat{\pi}(B) \end{array}
\end{tikzcd} \tag{8.15}$$

which translates the minor Clifford product into the major Clifford product according to

$$\hat{\pi}(A) * \hat{\pi}(B) := \hat{\pi}(AB) \qquad \forall\ A, B \in \Gamma_n \tag{8.16}$$

or with $A = \hat{\rho}(C)$ and $B = \hat{\rho}(D)$ according to

$$C * D := \hat{\pi}(\hat{\rho}(C)\hat{\rho}(D)) \qquad \forall\ C, D \in \Gamma_n. \tag{8.17}$$

The minor Clifford product, defined in equation (8.17), is associative,

$$(A * B) * C = \hat{\pi}(\hat{\rho}(A)\hat{\rho}(B)) * C = \hat{\pi}([\hat{\rho}(A)\hat{\rho}(B)]\, \hat{\rho}(C)) \tag{8.18}$$

$$\begin{aligned} &= \hat{\pi}(\hat{\rho}(A)\,[\hat{\rho}(B)\hat{\rho}(C)]) = A * \hat{\pi}(\hat{\rho}(B)\hat{\rho}(C)) \\ &= A * (B * C), \end{aligned}$$

distributive with respect to addition,

$$\begin{aligned} A * (B + C) &= \hat{\pi}(\hat{\rho}(A)[\hat{\rho}(B) + \hat{\rho}(C)]) \qquad \forall\ A, B, C \in \Gamma_n \\ &= \hat{\pi}(\hat{\rho}(A)\hat{\rho}(B)) + \hat{\pi}(\hat{\rho}(A)\hat{\rho}(C)) \\ &= A * B + A * C, \end{aligned} \tag{8.19}$$

$$\begin{aligned} (A + B) * C &= \hat{\pi}([\hat{\rho}(A) + \hat{\rho}(B)]\hat{\rho}(C)) \\ &= \hat{\pi}(\hat{\rho}(A)\hat{\rho}(C)) + \hat{\pi}(\hat{\rho}(B)\hat{\rho}(C)) \\ &= A * C + B * C, \end{aligned} \tag{8.20}$$

unital

$$\mathbf{1}^- := \hat{\pi}(\mathbf{1}^+) = \hat{\pi}(\mathbf{Z}^+) = \frac{1}{\lambda}\,\mathbf{Z}^- = \frac{1}{\lambda}\,\mathbf{I}^+, \tag{8.21}$$

$$\begin{aligned} (\mathbf{1}^-) * A &= \hat{\pi}(\mathbf{1}^+) * A = \hat{\pi}(\mathbf{1}^+\,\hat{\rho}(A)) = A = \hat{\pi}(\hat{\rho}(A)\,\mathbf{1}^+) \\ &= A * \hat{\pi}(\mathbf{1}^+) = A * (\mathbf{1}^-) \end{aligned} \tag{8.22}$$

and any scalar in the minus approach $X_{\overline{0}}$ commutes with any multivector M,

$$X_{\overline{0}} * M = M * X_{\overline{0}} = \alpha M \qquad \forall\ X_{\overline{0}} = \alpha\mathbf{1}^- \in \Lambda_n^{0-} \text{ and } \forall\ M \in \Lambda_n. \tag{8.23}$$

With equations (7.49) and (7.51) from Definition and Theorem 7.5 for generic 1-vectors in the minus approach $A, B \in \Lambda_n^{1-}$,

$$\begin{aligned} B_{\hat{\pi}}^{1+}(\hat{\rho}(A), \hat{\rho}(B))\mathbf{1}^+ &= B_{\hat{\pi}}^{1+}(\hat{\rho}(A), \hat{\rho}(B))\mathbf{Z}^+ \\ &= \frac{1}{2}(\hat{\rho}(A) \vee B + \hat{\rho}(B) \vee A)\,, \end{aligned} \tag{8.24}$$

$$\begin{aligned} \frac{1}{2}(\hat{\rho}(A) \wedge B + \hat{\rho}(B) \wedge A) &= (-1)^{n-1}\frac{1}{2}(B \wedge \hat{\rho}(A) + A \wedge \hat{\rho}(B)) \\ &= (-1)^{n-1}B_{\hat{\rho}}^{1-}(A, B)\mathbf{Z}^- \\ &= \lambda(-1)^{n-1}B_{\hat{\rho}}^{1-}(A, B)\mathbf{1}^- \end{aligned} \tag{8.25}$$

and get with Theorem 4.27

$$B_{\hat{\pi}}^{1+}(\hat{\rho}(A), \hat{\rho}(B)) = (-1)^{n-1}B_{\hat{\rho}}^{1-}(A, B). \tag{8.26}$$

Using equation (8.26), we get the dual decomposition rule for generic 1-vectors A and B in the minus approach,

$$\begin{aligned} A * B &= \hat{\pi}(\hat{\rho}(A)\hat{\rho}(B)) \\ &= \hat{\pi}\big(B_{\hat{\pi}}^{1+}(\hat{\rho}(A), \hat{\rho}(B))\mathbf{1}^+ + \hat{\rho}(A) \wedge \hat{\rho}(B)\big) \\ &= (-1)^{n-1}B_{\hat{\rho}}^{1-}(A, B)\mathbf{1}^- + \frac{1}{\mu}A \vee B \\ &= \frac{1}{\lambda}(-1)^{n-1}B_{\hat{\rho}}^{1-}(A, B)\mathbf{Z}^- + \frac{1}{\mu}A \vee B, \end{aligned} \tag{8.27}$$

and with it the dual contraction rule for generic 1-vectors A in the minus approach,

$$\begin{aligned} A^2 = A * A &= (-1)^{n-1} B_{\hat{\rho}}^{1-}(A,A)\mathbf{1}^- \qquad \forall\, A \in \Lambda_n^{1-} \qquad (8.28)\\ &= (-1)^{n-1} Q_{\hat{\rho}}^{1-}(A)\mathbf{1}^- = \frac{1}{\lambda}(-1)^{n-1} Q_{\hat{\rho}}^{1-}(A)\mathbf{Z}^-. \end{aligned}$$

In addition, using equation (8.26), for any orthogonal set

$$\mathcal{S}^- = \{X_1, X_2, \dots, X_m\} \subset \Lambda_n^{1-} \qquad (8.29)$$

of m homogeneous vectors of grade 1 in the minus approach, the set

$$\mathcal{S}^+ = \{\hat{\rho}(X_1), \hat{\rho}(X_2), \dots, \hat{\rho}(X_m)\} \subset \Lambda_n^{1+} \qquad (8.30)$$

of m homogeneous vectors of grade 1 in the plus approach is orthogonal too, and thus the minor Clifford and the minor exterior products are the same up to the factor $\frac{1}{\mu^{m-1}}$,

$$\mathop{\ast}_{l=1}^{m} X_l = \hat{\pi}\left(\prod_{l=1}^{m} \hat{\rho}(X_l)\right) = \hat{\pi}\left(\bigwedge_{l=1}^{m} \hat{\rho}(X_l)\right) = \frac{1}{\mu^{m-1}} \bigvee_{l=1}^{m} X_l. \qquad (8.31)$$

By requiring the major Clifford product to be associative in equation (8.11), to be distributive in equations (8.9) and (8.10), to have an one-element in equations (8.7) and (8.8), to satisfy the decomposition rule for generic 1-vectors in equation (8.12) and to reduce to the major outer product for any orthogonal set of 1-vectors in the plus approach according to equation (8.14), the former is defined.

In case of a non-degenerate quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$, a dual Clifford product, the minor Clifford product $*$, is defined by equation (8.17). Instead of using equation (8.17), we could define the minor Clifford product by requiring the latter to be associative as in equation (8.18), to be distributive as in equations (8.19) and (8.20), to have an one-element as in equations (8.21), (8.22) and (8.23), to satisfy the dual decomposition rule for generic 1-vectors as in equation (8.27) and to reduce to the minor outer product for any orthogonal set of 1-vectors in the minus approach according to equation (8.31). In case of a non-degenerate quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$ both ways are equivalent.

In case of the degenerate quadrics $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$, i. e. in case of a pair of naturally associated degenerate polarities $(\hat{\pi},\hat{\rho})$, the minor Clifford product cannot be defined by equation (8.17). For generic 1-vectors A, B in the minus approach, we would get for any naturally associated degenerate pair of polarities

$$\begin{aligned} A * B &= \hat{\pi}(\hat{\rho}(A)\hat{\rho}(B)) \qquad (8.32)\\ &= \frac{1}{\lambda}(-1)^{n-1} B_{\hat{\rho}}^{1-}(A,B)\,\hat{\pi}(\mathbf{Z}^-) + \frac{1}{\mu}\,\hat{\pi}\hat{\rho}(A \vee B)\\ &= \mathbf{0}. \end{aligned}$$

Since the dual contraction rule is not vanishing for all degenerate polarities $\hat{\rho}$, we cannot use equation (8.17) in the degenerate cases. Instead we define the minor Clifford product by requiring the latter to be associativ, distributive, to have an one-element, to satisfy the decomposition rule for generic 1-vectors in the minus approach and to reduce to the minor outer product for any orthogonal set of 1-vectors in the minus approach. This is done in

Definition 8.5 (Clifford Double Algebra Γ_n). Let $\Lambda_n(+,\cdot,\wedge,\vee)$ represent a projective $\mathbb{F}$-algebra[18] and let $(\hat{\pi},\hat{\rho})$ represent a pair of non-degenerate or degenerate naturally associated polarities[19] with $\lambda,\mu \in \{1,-1\}$. By definition, the *major Clifford product* (no sign) and the *minor Clifford product* $(*)$

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \longrightarrow & \Lambda_n \\ (A,B) & \longmapsto & AB \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{*} & \Lambda_n \\ (A,B) & \longmapsto & A*B \end{array} \tag{8.33}$$

meet the the following conditions:

(C1) Both Clifford products are associative and distributive with respect to addition in $\Lambda_n(+,\cdot)$.

(C2) The Clifford products between homogeneous 1-vectors decompose into a scalar and a 2-vector part according to,

$$AB := B^{1+}_{\hat{\pi}}(A,B)\mathbf{Z}^+ + A\wedge B \qquad \forall\ A,B \in \Lambda^{1+}_n, \tag{8.34}$$

$$A*B := \frac{1}{\lambda}(-1)^{n-1}B^{1-}_{\hat{\rho}}(A,B)\mathbf{Z}^- + \frac{1}{\mu}A\vee B \qquad \forall\ A,B \in \Lambda^{1-}_n, \tag{8.35}$$

where $B^{1+}_{\hat{\pi}}$ and $B^{1-}_{\hat{\rho}}$ represent the bilinear forms of the naturally associated pair $(\hat{\pi},\hat{\rho})$.

(C3) Both Clifford products are unital,

$$X_{\overline{0}}M = MX_{\overline{0}} := \alpha M \qquad \forall\ X_{\overline{0}} = \alpha\mathbf{1}^+ \in \Lambda^{0+}_n \text{ and } \forall\ M \in \Lambda_n, \tag{8.36}$$

$$X_{\overline{0}} * M = M * X_{\overline{0}} := \alpha M \qquad \forall\ X_{\overline{0}} = \alpha\mathbf{1}^- \in \Lambda^{0-}_n \text{ and } \forall\ M \in \Lambda_n, \tag{8.37}$$

$$\mathbf{1}^+ := \mathbf{Z}^+, \qquad \mathbf{1}^- := \hat{\pi}(\mathbf{1}^+) = \frac{1}{\lambda}\mathbf{Z}^-. \tag{8.38}$$

(C4) For an orthogonal set $\mathcal{S} = \{X_1, X_2, \ldots, X_m\} \subset \Lambda^1_n$ of m homogeneous vectors of grade 1 the Clifford and exterior products are the same,

$$\prod_{l=1}^{m} X_l := \bigwedge_{l=1}^{m} X_l \iff \left\{\begin{array}{l} \mathcal{S}^+ \text{ is a orthogonal} \\ \text{system with respect to} \\ \text{the bilinear form } B^{1+}_{\hat{\pi}}. \end{array}\right. \tag{8.39}$$

$$\mathop{\LARGE *}_{l=1}^{m} X_l := \frac{1}{\mu^{m-1}} \bigvee_{l=1}^{m} X_l \iff \left\{\begin{array}{l} \mathcal{S}^- \text{ is a orthogonal} \\ \text{system with respect to} \\ \text{the bilinear form } B^{1-}_{\hat{\rho}}. \end{array}\right. \tag{8.40}$$

The major and the minor Clifford products imprint the structure of a Clifford algebra twice onto the projective algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ with respect to the given pair $(\hat{\pi},\hat{\rho})$ of naturally associated polarities. With this additional structure the projective algebra $\Lambda_n(+,\cdot,\wedge,\vee)$ becomes a *Clifford double* $\mathbb{F}$-*algebra* $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ with signatures $S^+(s^+,t^+,u^+)$ and $S^-(s^-,t^-,u^-)$. We may also note it as $\Gamma_{s^+,t^+,u^+;s^-,t^-,u^-}$.

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[18] Compare Definition 5.1 and Definition 4.2.

[19] Compare Definition 7.21.

The major and minor Clifford products are well defined, since a) the conditions (C2) and (C4) do not interfere, b) the condition (C2) contains the contraction rule of a Clifford product displayed in equation (8.3) and c) it is unital, distributive and associative by definition.

Corollary 8.6. *Let* $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ *represent a Clifford double* $\mathbb{F}$*-algebra with signatures* $S^+(s^+,t^+,u^+)$ *and* $S^-(s^-,t^-,u^-)$ *according to Definition 8.5. The following properties result immediately:*

(D1) The plus approach to the above mentioned Clifford double algebra, i. e. $\Gamma_n^+ = \Gamma_{s^+,t^+,u^+}(+,\cdot,\ )$ *is a Clifford algebra with signature* $S^+(s^+,t^+,u^+)$*, the minus approach to the above mentioned Clifford double algebra, i. e.* $\Gamma_n^- = \Gamma_{s^-,t^-,u^-}(+,\cdot,*)$ *is a second Clifford algebra with signature* $S^-(s^-,t^-,u^-)$*, and both Clifford algebras share the same projective algebra* $\Lambda_n(+,\cdot,\wedge,\vee)$*.*

(D2) In case the pair of naturally associated polarities $(\hat\pi,\hat\rho)$ *is non-degenerate, there is an harmonic orthonormal system of bases* $\{P_{\mathbf{b}}\},\{E_{\mathbf{b}}\}\subset\Lambda_n$ *with the under (D1) mentioned signatures* S^+ *and* S^- *respectively and with*

$$Q_{\hat\pi}^{1+}(P_{\mathbf{b}})\cdot Q_{\hat\rho}^{1-}(E_{\mathbf{b}}) = \eta_{\mathbf{b}}^+\eta_{\mathbf{b}}^- = \frac{1}{\lambda\mu}. \tag{8.41}$$

(D3) In case the pair of naturally associated polarities $(\hat\pi,\hat\rho)$ *is degenerate, there is an harmonic orthonormal system of bases* $\{P_{\mathbf{b}}\},\{E_{\mathbf{b}}\}\subset\Lambda_n$ *with the under (D1) mentioned signatures* S^+ *and* S^- *respectively and with*

$$Q_{\hat\pi}^{1+}(P_{\mathbf{b}})\cdot Q_{\hat\rho}^{1-}(E_{\mathbf{b}}) = \eta_{\mathbf{b}}^+\eta_{\mathbf{b}}^- = 0. \tag{8.42}$$

Proof. Statement (D1) is a direkt consequence of Definition 8.5 about what a Clifford double algebra Γ_n is. Statements (D2) and (D3) follow immediately from Definition and Theorem 7.18 and Theorem 7.22 about harmonic orthonormal systems of bases respectively. □

Theorem 8.7. *Let*

$$P_{\mathbf{b}} = \mathbf{Z}^+, \qquad S(\mathbf{b}) = 0, \tag{8.43}$$

$$P_{\mathbf{b}} = \bigwedge_{l=1}^{S(\mathbf{b})} P_{{}_l\mathbf{b}} = \prod_{l=1}^{S(\mathbf{b})} P_{{}_l\mathbf{b}}, \qquad 1\le S(\mathbf{b})\le n, \tag{8.44}$$

and

$$E_{\mathbf{b}} = \mathbf{Z}^-, \qquad S(\mathbf{b}) = 0, \tag{8.45}$$

$$E_{\mathbf{b}} = \bigvee_{l=1}^{S(\mathbf{b})} E_{{}_l\mathbf{b}} = \mu^{S(\mathbf{b})-1}\cdot \mathop{\ast}\limits_{l=1}^{S(\mathbf{b})} E_{{}_l\mathbf{b}}, \qquad 1\le S(\mathbf{b})\le n, \tag{8.46}$$

represent an harmonic orthonormal system of bases of the Clifford double algebra Γ_n *and let the variables* $\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}$ *and* $\mathbf{u}$ *be* n*-digit binary numbers with*

$$\mathbf{d} = \mathbf{b}\ \text{AND}\ \mathbf{c}, \qquad \mathbf{e} = \mathbf{b}\ \text{XOR}\ \mathbf{c}, \qquad S(\mathbf{u}) = n. \tag{8.47}$$

*(D1) Two different basis-*1*-vectors* $P_{\mathbf{b}}$ *and* $P_{\mathbf{c}}$ *or* $E_{\mathbf{b}}$ *and* $E_{\mathbf{c}}$ *of the above mentioned harmonic orthonormal system of bases anticommute, two equal basis-*1*-vectors commute with respect to the Clifford products,*

$$P_{\mathbf{b}}P_{\mathbf{c}} = -P_{\mathbf{c}}P_{\mathbf{b}} \qquad \forall\ S(\mathbf{b}) = S(\mathbf{c}) = 1 \text{ and } \mathbf{c} \neq \mathbf{d} \tag{8.48}$$

$$E_{\mathbf{b}} * E_{\mathbf{c}} = -E_{\mathbf{c}} * E_{\mathbf{b}} \tag{8.49}$$

$$P_{\mathbf{b}}^2 = P_{\mathbf{b}}P_{\mathbf{b}} = Q_{\hat{\pi}}^{1+}(P_{\mathbf{b}})\mathbf{1}^+ = \eta_{\mathbf{b}}^+\mathbf{1}^+ \qquad \forall\ S(\mathbf{b}) = 1 \tag{8.50}$$

$$E_{\mathbf{b}}^2 = E_{\mathbf{b}} * E_{\mathbf{b}} = (-1)^{n-1}Q_{\hat{\rho}}^{1-}(E_{\mathbf{b}})\mathbf{1}^- = (-1)^{n-1}\eta_{\mathbf{b}}^-\mathbf{1}^- \tag{8.51}$$

(D2) The multiplication tables for the Clifford products with respect to the harmonic orthonormal system of bases $\{P_{\mathbf{b}}\}, \{E_{\mathbf{b}}\} \subset \Gamma_n$ *then are*

$$P_{\mathbf{b}}P_{\mathbf{c}} = \begin{cases} P_{\mathbf{b}}P_{\mathbf{c}}, & S(\mathbf{b}) = 0 \text{ or } S(\mathbf{c}) = 0, \\ \alpha_{\mathbf{bc}}P_{\mathbf{e}}, & S(\mathbf{d}) = 0, S(\mathbf{b}) \neq 0, S(\mathbf{c}) \neq 0, \\ \alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})} \eta_{{}_l\mathbf{d}}^+\right] P_{\mathbf{e}}, & S(\mathbf{d}) > 0, \end{cases} \tag{8.52}$$

$$E_{\mathbf{b}} * E_{\mathbf{c}} = \begin{cases} E_{\mathbf{b}}E_{\mathbf{c}}, & S(\mathbf{b}) = 0 \text{ or } S(\mathbf{c}) = 0, \\ \frac{1}{\mu}\alpha_{\mathbf{bc}}E_{\mathbf{e}}, & S(\mathbf{d}) = 0, S(\mathbf{b}) \neq 0, S(\mathbf{c}) \neq 0, \\ \mu^{S(\mathbf{b})+S(\mathbf{c})-S(\mathbf{e})-1}\alpha_{\mathbf{bc}}(-1)^{(n-1)S(\mathbf{d})}\left[\prod_{l=1}^{S(\mathbf{d})} \eta_{{}_l\mathbf{d}}^-\right] E_{\mathbf{e}}, & \\ & S(\mathbf{d}) > 0 \end{cases} \tag{8.53}$$

with

$$S(\mathbf{b}) + S(\mathbf{c}) = 2 \cdot S(\mathbf{d}) + S(\mathbf{e}) \tag{8.54}$$

$$0 \leq |S(\mathbf{b}) - S(\mathbf{c})| \leq S(\mathbf{e}) \leq \mathcal{D}_n(S(\mathbf{b}) + S(\mathbf{c})) \leq n \tag{8.55}$$

$$\mathcal{D}_n(i) = \begin{cases} i & 0 \leq i \leq n \\ 2n - i & n < i \leq 2n \end{cases} \tag{8.56}$$

and in the special case $\mathbf{c} = \mathbf{u}$, $\mathbf{I}^+ = P_{\mathbf{u}}$ *and* $\mathbf{I}^- = E_{\mathbf{u}}$

$$\begin{aligned} P_{\mathbf{b}}\mathbf{I}^+ &= (-1)^{(n-1)S(\mathbf{b})}\mathbf{I}^+P_{\mathbf{b}} \qquad (8.57) \\ &= \begin{cases} P_{\mathbf{b}}\mathbf{I}^+, & S(\mathbf{d}) = 0, \\ \alpha_{\mathbf{bu}}\left[\prod_{l=1}^{S(\mathbf{b})} \eta_{{}_l\mathbf{b}}^+\right] P_{\overline{\mathbf{b}}}, & S(\mathbf{d}) > 0, \end{cases} \\ E_{\mathbf{b}} * \mathbf{I}^- &= (-1)^{(n-1)S(\mathbf{b})}\mathbf{I}^- * E_{\mathbf{b}} \qquad (8.58) \\ &= \begin{cases} E_{\mathbf{b}}\mathbf{I}^-, & S(\mathbf{d}) = 0, \\ \mu^{2S(\mathbf{b})-1}\alpha_{\mathbf{bu}}(-1)^{(n-1)S(\mathbf{b})}\left[\prod_{l=1}^{S(\mathbf{b})} \eta_{{}_l\mathbf{b}}^-\right] E_{\overline{\mathbf{b}}}, & S(\mathbf{d}) > 0, \end{cases} \end{aligned}$$

Proof. Regarding statement (D1), equations (8.48) and (8.49) are a consequence from equations (8.39) and (8.40) respectively,

$$P_{\mathbf{b}}P_{\mathbf{c}} = P_{\mathbf{b}} \wedge P_{\mathbf{c}} = -P_{\mathbf{c}} \wedge P_{\mathbf{b}} = -P_{\mathbf{c}}P_{\mathbf{b}} \tag{8.59}$$

$$E_{\mathbf{b}} * E_{\mathbf{c}} = \frac{1}{\mu}\, E_{\mathbf{b}} \vee E_{\mathbf{c}} = -\frac{1}{\mu}\, E_{\mathbf{c}} \vee E_{\mathbf{b}} = -E_{\mathbf{c}} * E_{\mathbf{b}} \tag{8.60}$$

with $S(\mathbf{b}) = S(\mathbf{c}) = 1$ and $\mathbf{c} \neq \mathbf{d}$. Equations (8.50) and (8.51) follow from equations (8.34), (8.35) and (8.38) as well as with equations (7.65) and (7.67) respectively. Trivially, in case of two equal basis-1-vectors the Clifford products commute.

Regarding statement (D2), the Clifford products reduce in case of $S(\mathbf{b}) = 0$ or $S(\mathbf{c}) = 0$ to the scalar product or the product in the field $\mathbb{F}$. In all other cases we need to reorder the basis 1-vectors in the products $P_{\mathbf{b}}P_{\mathbf{c}}$ or $E_{\mathbf{b}} * E_{\mathbf{c}}$ such, that equal 1-vectors are direct neighbours. Compare the proof to Theorem 4.10. Just the evaluation of the reordered products is different to Theorem 4.10. If there are two equal basis 1-vectors, their Clifford products reduce according to equations (8.50) or (8.51) respectively to the numbers $\eta^{+}_{\mathbf{b}}\mathbf{1}^{+}$ or $(-1)^{n-1}\eta^{-}_{\mathbf{b}}\mathbf{1}^{-}$.

For $S(\mathbf{b}) \neq 0$ and $S(\mathbf{c}) \neq 0$ we get

$$\begin{aligned} P_{\mathbf{b}}P_{\mathbf{c}} &= \left[\prod_{l=1}^{S(\mathbf{b})} P_{{}_l\mathbf{b}}\right]\left[\prod_{l=1}^{S(\mathbf{c})} P_{{}_l\mathbf{c}}\right] = \alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})} (P_{{}_l\mathbf{d}})^2\right]\left[\prod_{l=1}^{S(\mathbf{e})} P_{{}_l\mathbf{e}}\right] \\ &= \alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{{}_l\mathbf{d}}\right] P_{\mathbf{e}}, \end{aligned} \tag{8.61}$$

$$\begin{aligned} E_{\mathbf{b}} * E_{\mathbf{c}} &= \left[\bigvee_{l=1}^{S(\mathbf{b})} E_{{}_l\mathbf{b}}\right] * \left[\bigvee_{l=1}^{S(\mathbf{c})} E_{{}_l\mathbf{c}}\right] \\ &= \mu^{S(\mathbf{b})-1}\mu^{S(\mathbf{c})-1}\left[\mathop{\ast}\limits_{l=1}^{S(\mathbf{b})} E_{{}_l\mathbf{b}}\right] * \left[\mathop{\ast}\limits_{l=1}^{S(\mathbf{c})} E_{{}_l\mathbf{c}}\right] \\ &= \mu^{S(\mathbf{b})-1}\mu^{S(\mathbf{c})-1}\alpha_{\mathbf{bc}}\left[\mathop{\ast}\limits_{l=1}^{S(\mathbf{d})} (E_{{}_l\mathbf{d}})^2\right] * \left[\mathop{\ast}\limits_{l=1}^{S(\mathbf{e})} E_{{}_l\mathbf{e}}\right] \\ &= \mu^{S(\mathbf{b})+S(\mathbf{c})-S(\mathbf{e})-1}\alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})} (-1)^{n-1}\eta^{-}_{{}_l\mathbf{d}}\right] * \left[\bigvee_{l=1}^{S(\mathbf{e})} E_{{}_l\mathbf{e}}\right] \\ &= \mu^{S(\mathbf{b})+S(\mathbf{c})-S(\mathbf{e})-1}\alpha_{\mathbf{bc}}(-1)^{(n-1)S(\mathbf{d})}\left[\prod_{l=1}^{S(\mathbf{d})} \eta^{-}_{{}_l\mathbf{d}}\right] E_{\mathbf{e}}. \end{aligned} \tag{8.62}$$

If $S(\mathbf{d}) = 0$, then $S(\mathbf{b}) = S(\mathbf{b}) + S(\mathbf{c})$ and $\prod_{l=1}^{S(\mathbf{d})} \eta^{\pm}_{{}_l\mathbf{d}} = 1$.

Equation (8.54) is a consequence of how the n-digit variables $\mathbf{d}$ and $\mathbf{e}$ are defined in equations (8.47). From equation (8.54) we get

$$S(\mathbf{b}) - S(\mathbf{c}) = 2\,[S(\mathbf{d}) - S(\mathbf{c})] + S(\mathbf{e}), \tag{8.63}$$

$$|S(\mathbf{b}) - S(\mathbf{c})| = |2\,[S(\mathbf{d}) - S(\mathbf{c})] + S(\mathbf{e})| \leq |S(\mathbf{e})|. \tag{8.64}$$

And if the sum $i = S(\mathbf{b}) + S(\mathbf{c})$ is greater than n, i.e. $n < i \leq 2n$, then the upper limit of $S(\mathbf{e})$ is $n - (i - n) = 2n - i$. This is why equations (8.55) and (8.56) are correct.

Equations (8.57) and (8.58) are special cases of equations (8.52) and (8.53) respectively. □

Notation 8.8 (Combined Clifford Product). Any mathematical term which contains the combined Clifford product $\circledast$ can be read twice: Firstly with respect to the plus approach as major Clifford product (no sign) and secondly with respect to the minus approach as minor Clifford product $*$.

Notation 8.9 (Multiple Combined Clifford Product Sign). For the multiple combined Clifford product we use the sign

$$\bigcircledast_{l=1}^{m} X_l := X_1 \circledast X_2 \circledast \cdots \circledast X_m. \tag{8.65}$$

Theorem 8.10. *For the Clifford product between the homogeneous multi vectors $X_{\overline{r}}$ and $Y_{\overline{s}}$ we get*

$$X_{\overline{r}} \circledast Y_{\overline{s}} = \sum_{k=0}^{m} \langle X_{\overline{r}} \circledast Y_{\overline{s}} \rangle_{|r-s|+2k} \tag{8.66}$$

with

$$m := \frac{\mathcal{D}_n(r+s) - |r-s|}{2} \tag{8.67}$$

and the index function $\mathcal{D}_n$ of equation (8.56).

Proof. We can write the homogeneous multi vectors $X_{\overline{r}}$ und $Y_{\overline{s}}$ as linear combinations with respect to the basis $B_{\mathbf{b}}$,

$$X_{\overline{r}} = \sum_{S(\mathbf{b})=r} \mu_{\mathbf{b}} B_{\mathbf{b}}, \qquad Y_{\overline{s}} = \sum_{S(\mathbf{c})=s} \nu_{\mathbf{c}} B_{\mathbf{c}}, \tag{8.68}$$

an then get

$$X_{\overline{r}} \circledast Y_{\overline{s}} = \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \mu_{\mathbf{b}} \nu_{\mathbf{c}} \, B_{\mathbf{b}} \circledast B_{\mathbf{c}}. \tag{8.69}$$

According to equations (8.52), (8.53) and (8.55) of Theorem 8.7, the grades of the product $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$ lie between $|r-s|$ and $\mathcal{D}_n(r+s)$ and differ by a multiple of 2 from the limits $|r-s|$ or $\mathcal{D}_n(r+s)$, since, if there are identical pairs present in the product $B_{\mathbf{b}} \circledast B_{\mathbf{c}}$, always two 1-vectors disappear. □

We are now in the position to compute the Clifford products of any two generic multi vectors M and N,

$$M \circledast N = \sum_{r,s=0}^{n} \langle M \rangle_r \circledast \langle N \rangle_s, \qquad M, N \in \Gamma_n. \tag{8.70}$$

Theorem 8.11. *Basis vectors $B_{\mathbf{b}}$ and $B_{\mathbf{c}}$ commute or anti-commute according to*

$$B_{\mathbf{b}} \circledast B_{\mathbf{c}} = (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})} B_{\mathbf{c}} \circledast B_{\mathbf{b}} \tag{8.71}$$

with the binary number $\mathbf{d} = \mathbf{b}$ AND $\mathbf{c}$.

Proof. The statement of Theorem 8.11 is evident in case the check sum of at least one of the binary indices is zero. This is why we assume $S(\mathbf{b}) \neq 0$ and $S(\mathbf{c}) \neq 0$ and get with Theorem 4.12

$$\begin{aligned} P_{\mathbf{b}}P_{\mathbf{c}} &= \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{{}_l\mathbf{d}}\right] P_{\mathbf{e}} = (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})}\alpha_{\mathbf{cb}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{{}_l\mathbf{d}}\right] P_{\mathbf{e}} \\ &= (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})} P_{\mathbf{c}}P_{\mathbf{b}} \end{aligned} \tag{8.72}$$

$$\begin{aligned} E_{\mathbf{b}} * E_{\mathbf{c}} &= (-1)^{S(\mathbf{d})(n-1)}\alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{-}_{{}_l\mathbf{d}}\right] E_{\mathbf{e}} \\ &= (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})}(-1)^{S(\mathbf{d})(n-1)}\alpha_{\mathbf{cb}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{-}_{{}_l\mathbf{d}}\right] E_{\mathbf{e}} \\ &= (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})} E_{\mathbf{c}} * E_{\mathbf{b}} \end{aligned} \tag{8.73}$$

□

8.1. Comparison with a Previous Definition of the Minor Clifford Product

For non-degenerate Clifford algebras only, a dual Clifford product was introduced in [Con08, pp. 16-18] by

$$A \,\hat{*}\, B := \left[A(\mathbf{I}^{+})^{-1}B(\mathbf{I}^{+})^{-1}\right](\mathbf{I}^{+})^{-1}. \tag{8.74}$$

It was called *dual geometric product* and the corresponding one-element is $(\mathbf{I}^{+})^{-1}$.

We will show in this subsection, how the definition of the dual geometric product $\hat{*}$ in equation (8.74) is related to the definition of the minor Clifford product $*$ in equation (8.17). In order to do so, we need the relations of

Corollary 8.12. *Let $\Gamma_n(+,\cdot,\wedge,\vee,\ ,*)$ represent a non-degenerate Clifford double $\mathbb{F}$-algebra with a pair of non-degenerate naturally associated polarities $(\hat{\pi},\hat{\rho})$, with $\lambda,\mu \in \{1,-1\}$ and with signatures $S^{+}(s^{+},t^{+},0)$ and $S^{-}(s^{-},t^{-},0)$. We are using the notations and the content of Theorem 8.7.*

We then have

$$\alpha_{\mathbf{bu}}\,\alpha_{\overline{\mathbf{b}}\mathbf{u}} = (-1)^{\frac{n(n-1)}{2}}, \tag{8.75}$$

$$\alpha_{\mathbf{b}\overline{\mathbf{b}}}\,\alpha_{\overline{\mathbf{b}}\mathbf{u}} = (-1)^{\frac{n(n-1)}{2}}(-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}}, \tag{8.76}$$

$$(-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}}(-1)^{\frac{S(\mathbf{c})(S(\mathbf{c})-1)}{2}} = (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})}(-1)^{\frac{S(\mathbf{e})(S(\mathbf{e})-1)}{2}} \tag{8.77}$$

Proof.

$$\begin{aligned} P_{\mathbf{b}} &= (-1)^{\frac{n(n-1)}{2}} \det\hat{\pi}\, P_{\mathbf{b}}\mathbf{I}^2 \\ &= (-1)^{\frac{n(n-1)}{2}} \det\hat{\pi}\,\alpha_{\mathbf{bu}} \left[\prod_{l=1}^{S(\mathbf{b})} \eta^{+}_{{}_l\mathbf{b}}\right] P_{\overline{\mathbf{b}}}\mathbf{I} \\ &= (-1)^{\frac{n(n-1)}{2}} \alpha_{\mathbf{bu}}\,\alpha_{\overline{\mathbf{b}}\mathbf{u}}\, P_{\mathbf{b}} \end{aligned} \tag{8.78}$$

$$\alpha_{\mathbf{b}\overline{\mathbf{b}}}\,(-1)^{\frac{n(n-1)}{2}}\det\hat{\pi} = \alpha_{\mathbf{b}\overline{\mathbf{b}}}\,\mathbf{I}^2 = P_{\mathbf{b}}P_{\overline{\mathbf{b}}}\mathbf{I} = \alpha_{\overline{\mathbf{b}}\mathbf{u}}\left[\prod_{l=1}^{S(\overline{\mathbf{b}})}\eta^{+}_{l\overline{\mathbf{b}}}\right]P^2_{\mathbf{b}} \tag{8.79}$$

$$= (-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}}\alpha_{\overline{\mathbf{b}}\mathbf{u}}\det\hat{\pi}$$

$$(-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}}(-1)^{\frac{S(\mathbf{c})(S(\mathbf{c})-1)}{2}}\left[\prod_{l=1}^{S(\mathbf{b})}\eta^{+}_{l\mathbf{b}}\right]\left[\prod_{l=1}^{S(\mathbf{c})}\eta^{+}_{l\mathbf{c}}\right] = \tag{8.80}$$

$$= P^2_{\mathbf{b}}P^2_{\mathbf{c}} = (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})}(P_{\mathbf{b}}P_{\mathbf{c}})^2 = (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})}P^2_{\mathbf{e}}$$

$$= (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})}(-1)^{\frac{S(\mathbf{e})(S(\mathbf{e})-1)}{2}}\left[\prod_{l=1}^{S(\mathbf{e})}\eta^{+}_{l\mathbf{e}}\right]$$

□

Let us now compute the minor Clifford products of two generic basis vectors $E_{\mathbf{b}}$ and $E_{\mathbf{c}}$ according to the definition in equation (8.17) and according to the definition in equation (8.74). We are using the notations and the content of Theorem 8.7 as well as Corollary 7.13 and Corollary 8.6.

$$E_{\mathbf{b}} * E_{\mathbf{c}} = \hat{\pi}(\hat{\rho}(E_{\mathbf{b}})\hat{\rho}(E_{\mathbf{c}})) = \hat{\gamma}_{\mathbf{bb}}\hat{\gamma}_{\mathbf{cc}}\,\hat{\pi}(P_{\mathbf{b}}P_{\mathbf{c}}) \tag{8.81}$$

$$= \hat{\gamma}_{\mathbf{bb}}\hat{\gamma}_{\mathbf{cc}}\alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})}\eta^{+}_{l\mathbf{d}}\right]\hat{\pi}(P_{\mathbf{e}})$$

$$= \hat{\gamma}_{\mathbf{bb}}\hat{\gamma}_{\mathbf{cc}}\alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})}\eta^{+}_{l\mathbf{d}}\right]\hat{\beta}_{\mathbf{ee}}\,E_{\mathbf{e}}$$

$$= (-1)^{S(\mathbf{b})(n-1)}\mu^{S(\mathbf{b})-1}\left[\prod_{l=1}^{S(\mathbf{b})}\eta^{-}_{l\mathbf{b}}\right]\cdot$$

$$\cdot\,(-1)^{S(\mathbf{c})(n-1)}\mu^{S(\mathbf{c})-1}\left[\prod_{l=1}^{S(\mathbf{c})}\eta^{-}_{l\mathbf{c}}\right]\cdot$$

$$\cdot\,\alpha_{\mathbf{bc}}\left[\prod_{l=1}^{S(\mathbf{d})}\eta^{+}_{l\mathbf{d}}\right]\lambda^{S(\mathbf{e})-1}\left[\prod_{l=1}^{S(\mathbf{e})}\eta^{+}_{l\mathbf{e}}\right]E_{\mathbf{e}}$$

$$= (-1)^{S(\mathbf{b})(n-1)}\mu^{S(\mathbf{b})-1}\left(\frac{1}{\lambda\mu}\right)^{S(\mathbf{b})}\left[\prod_{l=1}^{S(\mathbf{b})}\eta^{+}_{l\mathbf{b}}\right]\cdot$$

$$\cdot\,(-1)^{S(\mathbf{c})(n-1)}\mu^{S(\mathbf{c})-1}\left(\frac{1}{\lambda\mu}\right)^{S(\mathbf{c})}\left[\prod_{l=1}^{S(\mathbf{c})}\eta^{+}_{l\mathbf{c}}\right]\cdot$$

$$
\begin{aligned}
&\cdot \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] \lambda^{S(\mathbf{e})-1} \left[\prod_{l=1}^{S(\mathbf{e})} \eta^{+}_{l\mathbf{e}} \right] E_{\mathbf{e}} \\
&= (-1)^{S(\mathbf{b})(n-1)+S(\mathbf{c})(n-1)} \frac{\lambda^{S(\mathbf{e})}}{\lambda^{S(\mathbf{b})+S(\mathbf{c})+1}} \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] E_{\mathbf{e}} \\
&= (-1)^{S(\mathbf{b})(n-1)+S(\mathbf{c})(n-1)} \frac{\lambda^{S(\mathbf{e})+2S(\mathbf{d})}}{\lambda^{S(\mathbf{b})+S(\mathbf{c})+1}} \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] E_{\mathbf{e}} \\
&= \frac{1}{\lambda} (-1)^{S(\mathbf{b})(n-1)+S(\mathbf{c})(n-1)} \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] E_{\mathbf{e}}
\end{aligned}
$$

$$
\begin{aligned}
&E_{\mathbf{b}} \mathbin{\hat{*}} E_{\mathbf{c}} = \left[E_{\mathbf{b}} (\mathbf{I}^{+})^{-1} E_{\mathbf{c}} (\mathbf{I}^{+})^{-1} \right] (\mathbf{I}^{+})^{-1} = \\
&= E_{\mathbf{b}} \mathbf{I}^{+} E_{\mathbf{c}} = \alpha_{\mathbf{b}\overline{\mathbf{b}}} \alpha_{\mathbf{c}\overline{\mathbf{c}}} \, P_{\overline{\mathbf{b}}} \mathbf{I}^{+} P_{\overline{\mathbf{c}}} = (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \cdot \alpha_{\mathbf{b}\overline{\mathbf{b}}} \alpha_{\mathbf{c}\overline{\mathbf{c}}} \, P_{\overline{\mathbf{b}}} \mathbf{I}^{+} P_{\overline{\mathbf{c}}} (\mathbf{I}^{+})^{2} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \cdot \alpha_{\mathbf{b}\overline{\mathbf{b}}} \alpha_{\mathbf{c}\overline{\mathbf{c}}} \alpha_{\overline{\mathbf{b}}\mathbf{u}} \left[\prod_{l=1}^{S(\overline{\mathbf{b}})} \eta^{+}_{l\overline{\mathbf{b}}} \right] \alpha_{\overline{\mathbf{c}}\mathbf{u}} \left[\prod_{l=1}^{S(\overline{\mathbf{c}})} \eta^{+}_{l\overline{\mathbf{c}}} \right] P_{\mathbf{b}} P_{\mathbf{c}} \mathbf{I}^{+} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \, (-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}} (-1)^{\frac{S(\mathbf{c})(S(\mathbf{c})-1)}{2}} \left[\prod_{l=1}^{S(\mathbf{b})} \eta^{+}_{l\mathbf{b}} \right] \left[\prod_{l=1}^{S(\mathbf{c})} \eta^{+}_{l\mathbf{c}} \right] \cdot \\
&\quad \cdot \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] P_{\mathbf{e}} \mathbf{I}^{+} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \, (-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}} (-1)^{\frac{S(\mathbf{c})(S(\mathbf{c})-1)}{2}} \left[\prod_{l=1}^{S(\mathbf{b})} \eta^{+}_{l\mathbf{b}} \right] \left[\prod_{l=1}^{S(\mathbf{c})} \eta^{+}_{l\mathbf{c}} \right] \cdot \\
&\quad \cdot \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] \alpha_{\mathbf{eu}} \left[\prod_{l=1}^{S(\mathbf{e})} \eta^{+}_{l\mathbf{e}} \right] \alpha_{\mathbf{e}\overline{\mathbf{e}}} \, E_{\mathbf{e}} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \, (-1)^{\frac{S(\mathbf{b})(S(\mathbf{b})-1)}{2}} (-1)^{\frac{S(\mathbf{c})(S(\mathbf{c})-1)}{2}} (-1)^{\frac{S(\mathbf{e})(S(\mathbf{e})-1)}{2}} \cdot \\
&\quad \cdot \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] E_{\mathbf{e}} \\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \, (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})} \alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{l\mathbf{d}} \right] E_{\mathbf{e}}
\end{aligned}
\tag{8.82}
$$

From equation (8.82) we get with Theorem 8.11

$$
\begin{aligned}
&\alpha_{\mathbf{bc}} \left[\prod_{l=1}^{S(\mathbf{d})} \eta^{+}_{{}_l\mathbf{d}}\right] E_{\mathbf{e}} = && (8.83)\\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi}\, (-1)^{S(\mathbf{b})S(\mathbf{c})-S(\mathbf{d})} \left[E_{\mathbf{b}}(\mathbf{I}^{+})^{-1} E_{\mathbf{c}}(\mathbf{I}^{+})^{-1}\right] (\mathbf{I}^{+})^{-1}\\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi}\, \left[E_{\mathbf{c}}(\mathbf{I}^{+})^{-1} E_{\mathbf{b}}(\mathbf{I}^{+})^{-1}\right] (\mathbf{I}^{+})^{-1}\\
&= (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \cdot E_{\mathbf{c}} \mathbin{\hat{*}} E_{\mathbf{b}}.
\end{aligned}
$$

Inserting equation (8.83) into equation (8.81) we get

$$
\begin{aligned}
&E_{\mathbf{b}} * E_{\mathbf{c}} = \hat{\pi}(\hat{\rho}(E_{\mathbf{b}})\hat{\rho}(E_{\mathbf{c}})) && (8.84)\\
&= \frac{1}{\lambda}(-1)^{S(\mathbf{b})(n-1)+S(\mathbf{c})(n-1)}(-1)^{\frac{n(n-1)}{2}} \det \hat{\pi}\, \left[E_{\mathbf{c}}(\mathbf{I}^{+})^{-1} E_{\mathbf{b}}(\mathbf{I}^{+})^{-1}\right] (\mathbf{I}^{+})^{-1}\\
&= \frac{1}{\lambda}(-1)^{S(\mathbf{b})(n-1)+S(\mathbf{c})(n-1)}(-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \cdot E_{\mathbf{c}} \mathbin{\hat{*}} E_{\mathbf{b}}\\
&= \frac{1}{\lambda}(-1)^{\frac{(2[S(\mathbf{b})+S(\mathbf{c})]+n)(n-1)}{2}} \det \hat{\pi} \cdot E_{\mathbf{c}} \mathbin{\hat{*}} E_{\mathbf{b}}.
\end{aligned}
$$

We summarise the result in

Theorem 8.13. *Let* $X_{\overline{r}}$ *and* $Y_{\overline{s}}$ *represent two generic homogeneous multi vectors of a non-degenerate Clifford double algebra in the minus approach* Γ_n^-*. The minor Clifford product* $*$ *translates into the dual geometric product* $\hat{*}$ *defined in* [Con08, pp. 16-18] *as follows,*

$$
\begin{aligned}
X_{\overline{r}} * Y_{\overline{s}} &= \frac{1}{\lambda}(-1)^{\frac{(2[r+s]+n)(n-1)}{2}} \det \hat{\pi} \cdot \left[Y_{\overline{s}}(\mathbf{I}^{+})^{-1} X_{\overline{r}}(\mathbf{I}^{+})^{-1}\right] (\mathbf{I}^{+})^{-1} && (8.85)\\
&= \frac{1}{\lambda}(-1)^{\frac{(2[r+s]+n)(n-1)}{2}} \det \hat{\pi} \cdot Y_{\overline{s}} \mathbin{\hat{*}} X_{\overline{r}}.
\end{aligned}
$$

The one-element of the dual geometric product $\hat{*}$ *is* $(\mathbf{I}^{+})^{-\mathbf{1}}$ *and translates into the one-element of the minus approach to Clifford double algebra* Γ_n *according to*

$$
(\mathbf{I}^{+})^{-\mathbf{1}} = (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \cdot \mathbf{I}^{+} = \lambda(-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \cdot \mathbf{1}^{-} \qquad (8.86)
$$

Proof. Equation (8.85) is a consequence of equation (8.84) by representing the homogeneous multi vectors in terms of their basis vectors,

$$
X_{\overline{r}} = \sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} E_{\mathbf{b}} \qquad\qquad Y_{\overline{s}} = \sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} E_{\mathbf{c}}. \qquad (8.87)
$$

Equation (8.86) is a consequence of

$$
(\mathbf{I}^{+})^2 = (-1)^{\frac{n(n-1)}{2}} \det \hat{\pi} \qquad (8.88)
$$

and equations (8.38) and (4.151). ☐

8.2. Inner and Exterior Products in Terms of the Clifford Products

As it is well known from ordinary Clifford algebras Cl_n, the inner and exterior products of these algebras can be expressed in terms of the Clifford product. We will do the same here for the inner and exterior products of Clifford double algebras Γ_n. This also means, we will have to define the inner products in analogy to the inner product of an ordinary Clifford algebra Cl_n.

Let us first look at the relations of exterior and Clifford products in the framework of Clifford double algebra Γ_n.

Theorem 8.14. *For the homogeneous multi vectors $X_{\overline{r}}$ and $Y_{\overline{s}}$ we have*

$$X_{\overline{r}} \wedge Y_{\overline{s}} = \begin{cases} \langle X_{\overline{r}} Y_{\overline{s}} \rangle_{r+s}, & r \neq 0 \text{ and } s \neq 0, \\ \mathbf{0}, & r = 0 \text{ or } s = 0. \end{cases} \tag{8.89}$$

$$X_{\overline{r}} \vee Y_{\overline{s}} = \begin{cases} \mu \langle X_{\overline{r}} * Y_{\overline{s}} \rangle_{r+s}, & r \neq 0 \text{ and } s \neq 0, \\ \mathbf{0}, & r = 0 \text{ or } s = 0. \end{cases} \tag{8.90}$$

Proof. For $r = 0$ or $s = 0$ the exterior products $\Diamond$ vanishe by definition. Compare equations (4.6) and (4.7). And the Clifford products are $X_{\overline{r}} X_{\overline{s}}$, i. e. for $X_{\overline{r}} \neq \mathbf{0}$ and $X_{\overline{s}} \neq \mathbf{0}$ they do not vanish. In these cases, the exterior products $X_{\overline{r}} \Diamond Y_{\overline{s}}$ and the highest grade $r + s$ of the corresponding Clifford products $\langle X_{\overline{r}} \circledast Y_{\overline{s}} \rangle_{r+s}$ are different.

For $r \neq 0$ and $s \neq 0$ we compute the exterior products,

$$\begin{aligned} X_{\overline{r}} \Diamond Y_{\overline{s}} &= \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}} \right] \Diamond \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}} \right] \\ &= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \cdot B_{\mathbf{b}} \Diamond B_{\mathbf{c}} \\ &= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \alpha_{\mathbf{bc}} \delta_{0 S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \cdot B_{[\mathbf{b}\,\mathrm{XOR}\,\mathbf{c}]}, \end{aligned} \tag{8.91}$$

and the highest grade $r + s$ of the corresponding Clifford products,

$$\begin{aligned} \langle X_{\overline{r}} Y_{\overline{s}} \rangle_{r+s} &= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \, \langle P_{\mathbf{b}} P_{\mathbf{c}} \rangle_{r+s}. \\ &= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \alpha_{\mathbf{bc}} \delta_{0 S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \cdot P_{[\mathbf{b}\,\mathrm{XOR}\,\mathbf{c}]}. \end{aligned} \tag{8.92}$$

$$\begin{aligned} \mu \langle X_{\overline{r}} * Y_{\overline{s}} \rangle_{r+s} &= \mu \cdot \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \, \langle E_{\mathbf{b}} * E_{\mathbf{c}} \rangle_{r+s}. \\ &= \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}} \mu_{\mathbf{c}} \alpha_{\mathbf{bc}} \delta_{0 S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c})} \cdot B_{[\mathbf{b}\,\mathrm{XOR}\,\mathbf{c}]}. \end{aligned} \tag{8.93}$$

Thus, for $r \neq 0$ and $s \neq 0$, the two expressions $X_{\overline{r}} \wedge Y_{\overline{s}}$ and $\langle X_{\overline{r}} Y_{\overline{s}} \rangle_{r+s}$ as well as $X_{\overline{r}} \vee Y_{\overline{s}}$ and $\mu \langle X_{\overline{r}} * Y_{\overline{s}} \rangle_{r+s}$ are the same. Please also note, for $r+s > n$ both pairs vanish, since $S(\mathbf{d}) = S(\mathbf{b}\,\mathrm{AND}\,\mathbf{c}) > 0$. □

Definition 8.15 (Inner Products). Let $\Gamma_n(+, \cdot, \wedge, \vee, \ , *)$ represent a Clifford double algebra. The operations

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \overset{\cdot}{\longrightarrow} & \Lambda_n \\ (A,B) & \longmapsto & A \cdot B \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \overset{\circ}{\longrightarrow} & \Lambda_n \\ (A,B) & \longmapsto & A \circ B \end{array} \tag{8.94}$$

are called *major inner product* $(\cdot)$ and *minor inner product* $(\circ)$ and obey the following conditions:

$$X_{\overline{r}} \cdot Y_{\overline{s}} := \langle X_{\overline{r}} Y_{\overline{s}} \rangle_{|r-s|}, \qquad X_{\overline{r}} \in \Gamma_n^{r+},\ Y_{\overline{s}} \in \Gamma_n^{s+}, \tag{8.95}$$

$$X_{\overline{r}} \circ Y_{\overline{s}} := \langle Y_{\overline{r}} * X_{\overline{s}} \rangle_{|r-s|}, \qquad X_{\overline{r}} \in \Gamma_n^{r-},\ Y_{\overline{s}} \in \Gamma_n^{s-}, \tag{8.96}$$

$$A \cdot (B+C) = A \cdot B + A \cdot C, \qquad A, B, C \in \Gamma_n, \tag{8.97}$$

$$(A+B) \cdot C = A \cdot C + B \cdot C, \tag{8.98}$$

$$A \circ (B+C) = A \circ B + A \circ C, \tag{8.99}$$

$$(A+B) \circ C = A \circ C + B \circ C. \tag{8.100}$$

Please note, the defining equations (8.95) and (8.96) are also valid in the cases $r = 0$ or $s = 0$.

The inner products $\cdot$ and $\circ$ are well defined, since the corresponding Clifford products are distributive too.

Notation 8.16 (Combined Inner Product). Any mathematical term, which contains the combined inner product $\odot$ can be read twice: Firstly with respect to the plus approach as major inner product $\cdot$ and secondly with respect to the minus approach as minor inner product $\circ$.

Theorem 8.17. *In case of $r = 0$ or $s = 0$ the inner products, the Clifford products respectively and the scalar multiplication are the same,*

$$A_{\overline{r}} \odot B_{\overline{s}} = A_{\overline{r}} \circledast B_{\overline{s}} = A_{\overline{r}} B_{\overline{s}}, \qquad r = 0 \quad \text{or} \quad s = 0. \tag{8.101}$$

Proof. Compare equations (8.95) and (8.96) with equations (8.52) and (8.53) respectively. □

Theorem 8.18. *For homogeneous multi vectors of grade r and s with $r \geq s$ we have,*

$$X_{\overline{r}} \odot Y_{\overline{s}} = (-1)^{s(r-1)}\, Y_{\overline{s}} \odot X_{\overline{r}}. \tag{8.102}$$

Proof. Let $\mathbf{b}$ and $\mathbf{c}$ be n-digit binary numbers with sum of digits $S(\mathbf{b}) = r$ and $S(\mathbf{c}) = s$. Then there are coefficients $\lambda_{\mathbf{b}}$ and $\mu_{\mathbf{c}}$ such that

$$X_{\overline{r}} \odot Y_{\overline{s}} = \left\langle \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}} \right] \circledast \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}} \right] \right\rangle_{r-s} \tag{8.103}$$

$$
\begin{aligned}
&= \left\langle \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot B_{\mathbf{b}} \circledast B_{\mathbf{c}} \right\rangle_{r-s} \\
&= \left\langle \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} \lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot \alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}}\, B_{\mathbf{c}} \circledast B_{\mathbf{b}} \right\rangle_{r-s} \\
&= \left\langle \sum_{\substack{S(\mathbf{b}) = r \\ S(\mathbf{c}) = s}} (-1)^{rs-s}\lambda_{\mathbf{b}}\mu_{\mathbf{c}} \cdot B_{\mathbf{c}} \circledast B_{\mathbf{b}} \right\rangle_{r-s} \\
&= \left\langle (-1)^{rs-s} \cdot \left[\sum_{S(\mathbf{c})=s} \mu_{\mathbf{c}} B_{\mathbf{c}} \right] \circledast \left[\sum_{S(\mathbf{b})=r} \lambda_{\mathbf{b}} B_{\mathbf{b}} \right] \right\rangle_{r-s} \\
&= (-1)^{s(r-1)} \cdot Y_{\bar{s}} \odot X_{\bar{r}}.
\end{aligned}
$$

where we used Theorem 8.11 and equation (4.43). □

Theorem 8.19. *For generic* 1*-vectors* $X_{\bar{1}} = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} B_{\mathbf{b}} \in \Lambda_n^1$ *and* $Y_{\bar{1}} = \sum_{S(\mathbf{c})=1} \mu_{\mathbf{c}} B_{\mathbf{c}} \in \Lambda_n^1$ *we have*

$$X_{\bar{1}} Y_{\bar{1}} = X_{\bar{1}} \cdot Y_{\bar{1}} + X_{\bar{1}} \wedge Y_{\bar{1}}, \tag{8.104}$$

$$X_{\bar{1}} * Y_{\bar{1}} = X_{\bar{1}} \circ Y_{\bar{1}} + \frac{1}{\mu} X_{\bar{1}} \vee Y_{\bar{1}}. \tag{8.105}$$

The inner products are symmetric with respect to 1*-vectors,*

$$
\begin{aligned}
X_{\bar{1}}^{+} \cdot Y_{\bar{1}}^{+} &= B_{\hat{\pi}}^{1+}(X_{\bar{1}}^{+}, Y_{\bar{1}}^{+})\mathbf{Z}^{+} = \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}\mu_{\mathbf{b}}\hat{\beta}_{\mathbf{bb}}\mathbf{Z}^{+} && (8.106) \\
&= \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}\mu_{\mathbf{b}}\eta_{\mathbf{b}}^{+}\, \mathbf{1}^{+}, \\
X_{\bar{1}}^{-} \circ Y_{\bar{1}}^{-} &= \frac{1}{\lambda}(-1)^{n-1} B_{\hat{\rho}}^{1-}(X_{\bar{1}}^{-}, Y_{\bar{1}}^{-})\mathbf{Z}^{-} && (8.107) \\
&= \frac{1}{\lambda} \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}\mu_{\mathbf{b}}\hat{\gamma}_{\mathbf{bb}}\mathbf{Z}^{-} = (-1)^{n-1} \sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}}\mu_{\mathbf{b}}\eta_{\mathbf{b}}^{-}\, \mathbf{1}^{-},
\end{aligned}
$$

the exterior products are antisymmetric,

$$X_{\bar{1}} \Diamond Y_{\bar{1}} = \sum_{S(\mathbf{b})=2} \left(\lambda_{[{}_1\mathbf{b}]}\mu_{[{}_2\mathbf{b}]} - \lambda_{[{}_2\mathbf{b}]}\mu_{[{}_1\mathbf{b}]} \right) B_{\mathbf{b}}. \tag{8.108}$$

Proof. Equations (8.104) and (8.105) are the equations (8.34) and (8.35) from Definition 8.5. According to equation (4.12) the exterior product between two 1-vectors is antisymmetric. The antisymmetry is confirmed by expression (8.108).

The inner products between 1-vectors are clearly symmetric. Compare equations (8.106) and (8.107). □

Theorem 8.20 (Graßmann Algebras as Most Degenerate Clifford Algebras). *The Clifford double algebra* $\Gamma_n \equiv \Gamma_{0,0,n;0,0,n}$ *with signatures*

$$S^+(0,0,n) \qquad\qquad S^-(0,0,n) \tag{8.109}$$

and the parameters

$$\lambda = \mu = 1 \tag{8.110}$$

represents a Graßmann double algebra, i. e. $\Gamma_n^+(+,\cdot,\ )$ *and* $\Gamma_n^-(+,\cdot,*)$ *are both ordinary Graßmann algebras. The one-element of* $\Gamma_n^+(+,\cdot,\ )$ *is* $\mathbf{1}^+ = \mathbf{Z}^+$*, the one-element of* $\Gamma_n^-(+,\cdot,*)$ *is* $\mathbf{1}^- = \mathbf{Z}^-$*,*

$$\mathbf{1}^+ A = A\mathbf{1}^+ = A \qquad \forall\ A \in \Gamma^+_{(0,0,n)}, \tag{8.111}$$

$$\mathbf{1}^- * A = A * \mathbf{1}^- = A \qquad \forall\ A \in \Gamma^-_{(0,0,n)}, \tag{8.112}$$

while these elements continue to be zero divisors with respect to the exterior products,

$$\mathbf{Z}^+ \wedge A = A \wedge \mathbf{Z}^+ = \mathbf{0} \qquad \forall\ A \in \Gamma^+_{(0,0,n)}, \tag{8.113}$$

$$\mathbf{Z}^- \vee A = A \vee \mathbf{Z}^- = \mathbf{0} \qquad \forall\ A \in \Gamma^-_{(0,0,n)}. \tag{8.114}$$

Graßmann algebras are most degenerate Clifford algebras and the unital property is the last remnant of the Clifford product. For all non-zero grades the Clifford product and the respective exterior product are the same.

Proof. According to equations (8.36), (8.37) and (8.38) of Definition 8.5 $\mathbf{Z}^+$ and $\mathbf{Z}^-$ are the respective identity elements in the both approaches. And with the vanishing quadratic forms, the Clifford products reduce to the outer products for all non-zero grades,

$$X_{\overline{r}} \circledast Y_{\overline{s}} = X_{\overline{r}} \Diamond Y_{\overline{s}} \qquad \text{for all } r \neq 0, s \neq 0. \tag{8.115}$$

Compare also equations (8.52) and (8.53). □

8.3. Metric Geometries

While, from section 5, on projective geometry $\mathcal{P}_n$ was expressed in terms of projective algebra Λ_n, we can now describe the metric geometries in terms of Clifford double algebras Γ_n. It is well known that metric Cayley-Klein geometries can be expressed in terms of Clifford algebras. [Gun11a, Kla14] The new aspect here in this approach is to provide Clifford *double* algebras Γ_n as the natural tool to describe the metric geometries with their inherent dual structures or what is left of it in the degenerate cases.

Charles Gunn developped in his PhD-thesis [Gun11a], how metric Cayley-Klein spaces are "created in projective space $\mathbb{R}P^n$ based on quadratic forms" [Gun11a, p. 31] and Clifford algebras, i. e. he showed how to do the Caley-Klein contruction with Clifford algebras. Later on he called these Clifford algebras *projective geometric algebras* or abbreviated PGA. [Gun17] Our approach here to metric geometries is the same, except for we are using Clifford double algebras Γ_n instead of single Clifford algebras Cl_n with just one Clifford product.

Gunn introduced the notation $\mathbf{P}(\mathbb{R}^*_{3,0,1})$ for the 3D euclidean PGA. LEO DORST and STEVEN DE KENINCK [DDK24] use the term PGA as abbreviation for "plane-based geometric algebra" and denote the 3D euclidean PGA by $\mathbb{R}_{3,0,1}$. We will show in this section, how with the Clifford *double* algebra $\Gamma_{1,0,3;3,0,1}$ a missing, but highly degenerate part of 3D euclidean geometry becomes available through the second Clifford product.

Definition 8.21 (Metric Geometries $\mathcal{M}_n$). A *metric geometry* $\mathcal{M}_n$ is a Cayley-Klein geometry. It is defined in the context of projective geometry $\mathcal{P}_n$ by singling out an absolute quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$ with respect to a pair of naturally associated polarities $(\hat{\pi}, \hat{\rho})$. The metric geometry $\mathcal{M}_n$ is called *non-degenerate* or *degenerate* depending on whether the absolute quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$ and its pair of naturally associated polarities $(\hat{\pi}, \hat{\rho})$ are non-degenerate or degenerate respectively. The respective Clifford double algebra Γ_n provides the language to describe metric geometry $\mathcal{M}_n$.

Compared with projective geometry $\mathcal{P}_n$, in which metric geometry $\mathcal{M}_n$ is embedded, the latter provides additional instruments and structure with respect to the absolute quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$ such as orthogonality or metric concepts like distance and angle. These additional instruments and structure are expressed in terms of the respective Clifford double algebra Γ_n.

8.4. Metric Geometries. Space and Counterspace

The projective and thus universal version of the concept of space and counterspace was given in Definition 5.6. We will specialise this concept here to its metric version.

Definition 8.22 (Space and Counterspace. Metric Version). Inasmuch as metric geometry $\mathcal{M}_n$ is expressed in terms of the plus approach Γ_n^+ it is called *space* or *metric space* and inasmuch as it is expressed in terms of the minus approach Γ_n^- it is called *counterspace* or *metric counterspace.*

Metric space and metric counterspace contain projective space and projective counterspace respectively, since metric geometry $\mathcal{M}_n$ is embedded in projective geometry $\mathcal{P}_n$.[20]

8.5. Metric Geometries. Euclidean and Dual Euclidean Geometry

In his paper *On the Homogeneous Model of Euclidean Geometry* [Gun11b, Gun11c], CHARLES GUNN introduces first the projectivised exterior algebra $W = \mathbf{P}(\bigwedge \mathbb{R}^{n+1})$ and the projectivised dual exterior algebra $W^* = \mathbf{P}(\bigwedge \mathbb{R}^{(n+1)*})$, grants 'equal rights for W and W^*'[21] with respect to the task to represent the real projective space $\mathbb{R}P^n$ and refers 'to W as a *point-based* and W^* as a *plane-based* algebra.'[22]

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[20] An earlier version, of what the terms *space* and *counterspace* mean, was given in Definition 10 of [Con08, p. 71]. The latter is now replaced by the Definitions 5.6 and 8.22 of this article.

[21] [Gun11c, Sect. 2.2]

[22] [Gun11c, Sect. 2.2]

He then continues to Clifford algebra. 'It is fully determined by its signature, which describes the inner product structure. The signature is a triple of integers (p, n, z) where $p+n+z$ is the dimension of the underlying vector space, and p, n, and z are the numbers of positive, negative, and zero entries along the diagonal of the quadratic form representing the inner product. We denote the corresponding Clifford algebra constructed on the point-based Graßmann algebra as $\mathbf{P}(\mathbb{R}_{p,n,z})$; that based on the plane-based Graßmann algebra, as $\mathbf{P}(\mathbb{R}^*_{p,n,z})$.

The discovery and application of signatures to create different sorts of metric spaces within projective space goes back to a technique invented by ARTHUR CAYLEY and developed by FELIX KLEIN [Kle28]. The so-called *Cayley-Klein construction* provides models of the three standard metric geometries (hyperbolic, elliptic, and euclidean) – along with many others! – within projective space. This work provides the mathematical foundation for the inner product as it appears within the homogeneous model of Clifford algebra.'[23]

CHARLES GUNN works out the signatures for the euclidean inner products and concludes: 'As noted above, the euclidean inner product has signature $(1, 0, 3)$ on points and $(3, 0, 1)$ on planes. If we attach the first signature to W, we have the following relations for the basis 1-vectors:

$$(\mathbf{e}^0)^2 = 1; \qquad (\mathbf{e}^1)^2 = (\mathbf{e}^2)^2 = (\mathbf{e}^3)^2 = 0. \tag{8.116}$$

It's easy to see that these relations imply that, for all basis trivectors $\mathbf{E}_i$, $\mathbf{E}_i^2 = 0$. But the trivectors represent planes, and the signature for the plane-wise euclidean metric is $(3, 0, 1)$, not $(0, 0, 4)$. Hence, we cannot use W to arrive at euclidean space. If instead, we begin with W^*, and attach the plane-wise signature $(3, 0, 1)$, we obtain:

$$(\mathbf{e}_0)^2 = 0; \qquad (\mathbf{e}_1)^2 = (\mathbf{e}_2)^2 = (\mathbf{e}_3)^2 = 1. \tag{8.117}$$

It is easy to check that this inner product, when extended to the higher grades, produces the proper behaviour on the trivectors, since only $\mathbf{E}_0 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ has non-zero square, producing the point-wise signature $(0, 1, 3)$ (equivalent to the signature $(1, 0, 3)$). Hence, W^* is the correct choice for constructing a model of euclidean geometry.'[24]

The relation of the exterior algebras W and W^* to projective algebra Λ_n, and the relation of the Clifford algebra constructed on the point-based Graßmann algebra $\mathbf{P}(\mathbb{R}_n)$ as well as the Clifford algebra constructed on the plane-based Graßmann algebra $\mathbf{P}(\mathbb{R}^*_n)$ to Clifford double algebra Γ_n will be illustrated using 3D euclidean geometry as an example. Except for the Graßmann algebras W and W^* do have an one-element each, but projective algebra in the plus approach Λ_4^+ as well as in the minus approach do not have one-elements, the projectivised exterior algebra $W = \mathbf{P}(\bigwedge \mathbb{R}^4)$ and the projective algebra in the plus approach Λ_4^+ as well as the projectivised dual

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[23] [Gun11c, Sect. 3]

[24] [Gun11c, Sect. 3.2]

exterior algebra $W^* = \mathbf{P}(\bigwedge \mathbb{R}^{4*})$ and the projective algebra in the minus approach Λ_4^- are isomorph,

$$W(+,\ ,\vee) = \mathbf{P}(\bigwedge \mathbb{R}^4) \ \cong\ \Lambda_4^+(+,\ ,\wedge), \tag{8.118}$$

$$W^*(+,\ ,\wedge) = \mathbf{P}(\bigwedge \mathbb{R}^{4*}) \ \cong\ \Lambda_4^-(+,\ ,\vee). \tag{8.119}$$

Please note the opposite sign conventions for the exterior products. While Charles Gunn is using the join $\vee$ in the point-based algebra W, we are using the major exterior product sign $\wedge$ in the plus approach and denote the operation of connecting points with it. And while Gunn is using the meet $\wedge$ in the plane-based algebra W^*, we are using the minor exterior product sign $\vee$ in the minus approach and denote the operation of intersecting planes with it. This is *not* a conceptual difference. We just use different sign conventions.

The corresponding bases vector notations are displayed in Table 16, extending Table 1 of [Gun11c, p. 5].

Feature		Λ_4^+	W	W^*	Λ_4^-
0-vector	$S(\mathbf{b}) = 0$	$P_\mathbf{b} = \mathbf{Z}^+$	scalar $\mathbf{1}^0$	scalar $\mathbf{1}_0$	$E_\mathbf{b} = \mathbf{Z}^-$
1-vector	$S(\mathbf{b}) = 1$	$\{P_\mathbf{b}\}$	point $\{\mathbf{e}^i\}$	plane $\{\mathbf{e}_i\}$	$\{E_\mathbf{b}\}$
2-vector	$S(\mathbf{b}) = 2$	$\{P_\mathbf{b}\}$	spear $\{\mathbf{e}^{ij}\}$	axis $\{\mathbf{e}_{ij}\}$	$\{E_\mathbf{b}\}$
3-vector	$S(\mathbf{b}) = 3$	$\{P_\mathbf{b}\}$	plane $\{\mathbf{E}^i\}$	point $\{\mathbf{E}_i\}$	$\{E_\mathbf{b}\}$
4-vector	$S(\mathbf{b}) = 4$	$P_\mathbf{b} = \mathbf{I}^+$	$\mathbf{I}^0$	$\mathbf{I}_0$	$E_\mathbf{b} = \mathbf{I}^-$
outer product		$\wedge$	join $\vee$	meet $\wedge$	$\vee$

Table 16. Comparison of Λ_4^+, W, W^* and Λ_4^-.

The point-based Clifford algebra $\mathbf{P}(\mathbb{R}_{1,0,3})$ is isomorph to the plus approach of the Clifford double algebra $\Gamma_{1,0,3;3,0,1}(+,\ ,\wedge,\vee,\ ,*)$ and the plane-based Clifford algebra $\mathbf{P}(\mathbb{R}^*_{3,0,1})$ is isomorph to the minus approach of the same degenerate Clifford double algebra,

$$\mathbf{P}(\mathbb{R}_{1,0,3})(+,\ ,\vee,\ ) \cong \Gamma^+_{1,0,3;3,0,1}(+,\ ,\wedge,\ ), \tag{8.120}$$

$$\mathbf{P}(\mathbb{R}^*_{3,0,1})(+,\ ,\wedge,\ ) \cong \Gamma^-_{1,0,3;3,0,1}(+,\ ,\vee,*). \tag{8.121}$$

We choose our bases 1-vectors in the following way,

$$P_{0001} = \mathbf{e}^1, \qquad P_{0010} = \mathbf{e}^2, \qquad P_{0100} = \mathbf{e}^3, \qquad P_{1000} = \mathbf{e}^0, \tag{8.122}$$

$$E_{0001} = \mathbf{e}_1, \qquad E_{0010} = \mathbf{e}_2, \qquad E_{0100} = \mathbf{e}_3, \qquad E_{1000} = \mathbf{e}_0, \tag{8.123}$$

with

$$P^2_{0001} = P^2_{0010} = P^2_{0100} = \mathbf{0}, \qquad P^2_{1000} = \mathbf{Z}^+, \tag{8.124}$$

$$E^2_{0001} = E^2_{0010} = E^2_{0100} = \mathbf{Z}^-, \qquad E^2_{1000} = \mathbf{0}, \tag{8.125}$$

and

$$\mu = -\lambda = 1, \qquad \mathbf{1}^+ = \mathbf{Z}^+, \qquad \mathbf{1}^- = -\mathbf{Z}^-. \tag{8.126}$$

In equations (8.120) and (8.121) the opposite sign conventions with respect to the exterior products show up again, now in the context of the metric Clifford algebras. And while Charles Gunn is not introducing two different notations for the two different Clifford products of $\mathbf{P}(\mathbb{R}_{1,0,3})$ and $\mathbf{P}(\mathbb{R}^*_{3,0,1})$, we use juxtaposition (no sign) for the major Clifford product and $*$ for the minor Clifford product.

The outcome of Charles Gunn's study on the euclidean inner product was: It 'has signature $(1,0,3)$ on points and $(3,0,1)$ on planes.' [Gun11c, Sect. 3.2]. In the degenerate Clifford double algebra $\Gamma_{1,0,3;3,0,1}$ these signatures show up in the plus and in the minus approach respectively. We do not have to reject the point-wise signature of W, since there are two different Clifford products available in the double algebra and each of the products carries one of the two signatures.

The details of the 'homogeneous model of euclidean geometry' [Gun11b] expressed in terms of the Clifford double algebra $\Gamma_{1,0,3;3,0,1}$ with $n = 4$ and $\mu = -\lambda = 1$ are: The signatures in the plus and minus approach,

$$S^+(1,0,3): \qquad \eta^+_{0001} = \eta^+_{0010} = \eta^+_{0100} = 0 \qquad \eta^+_{1000} = 1, \tag{8.127}$$

$$S^-(3,0,1): \qquad \eta^-_{0001} = \eta^-_{0010} = \eta^-_{0100} = 1 \qquad \eta^-_{1000} = 0; \tag{8.128}$$

the matrices representing the pair of naturally associated polarities $(\hat{\pi}, \hat{\rho})$ of the absolute quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$,

$$\hat{B}_{\overline{0}} = (0)\,, \qquad \hat{\Gamma}_{\overline{0}} = (0)\,, \tag{8.129}$$

$$\hat{B}_{\overline{1}} = \mathrm{diag}(0,0,0,1), \qquad \hat{\Gamma}_{\overline{1}} = \mathrm{diag}(-1,-1,-1,0), \tag{8.130}$$

$$\hat{B}_{\overline{2}} = \mathrm{diag}(0,0,0,0,0,0), \qquad \hat{\Gamma}_{\overline{2}} = \mathrm{diag}(1,1,1,0,0,0), \tag{8.131}$$

$$\hat{B}_{\overline{3}} = \mathrm{diag}(0,0,0,0), \qquad \hat{\Gamma}_{\overline{3}} = \mathrm{diag}(-1,0,0,0), \tag{8.132}$$

$$\hat{B}_{\overline{4}} = (0)\,, \qquad \hat{\Gamma}_{\overline{4}} = (0)\,; \tag{8.133}$$

the polarity mappings $\hat{\pi}$ and $\hat{\rho}$ themselves,

$$\begin{array}{llcl}
\hat{\pi}: & \Lambda_n^+ & \longrightarrow & \Lambda_n^- \\
& X_{\overline{1}} = \displaystyle\sum_{S(\mathbf{b})=1} \lambda_{\mathbf{b}} P_{\mathbf{b}} & \longmapsto & \hat{\pi}(X_{\overline{1}}) = \lambda_{1000} E_{1000} \\
& X_{\overline{2}} = \displaystyle\sum_{S(\mathbf{b})=2} \lambda_{\mathbf{b}} P_{\mathbf{b}} & \longmapsto & \hat{\pi}(X_{\overline{2}}) = \mathbf{0} \\
& X_{\overline{3}} = \displaystyle\sum_{S(\mathbf{b})=3} \lambda_{\mathbf{b}} P_{\mathbf{b}} & \longmapsto & \hat{\pi}(X_{\overline{3}}) = \mathbf{0}
\end{array} \tag{8.134}$$

$$\begin{array}{llcl}
\hat{\rho}: & \Lambda_n^- & \longrightarrow & \Lambda_n^+ \\
& X_{\overline{1}} = \displaystyle\sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}} E_{\mathbf{b}} & \longmapsto & \hat{\rho}(X_{\overline{1}}) = -\mu_{0001} P_{0001} - \mu_{0010} P_{0010} - \\
& & & \qquad - \mu_{0100} P_{0100} \\
& X_{\overline{2}} = \displaystyle\sum_{S(\mathbf{b})=2} \mu_{\mathbf{b}} E_{\mathbf{b}} & \longmapsto & \hat{\rho}(X_{\overline{2}}) = +\mu_{0011} P_{0011} + \mu_{0101} P_{0101} + \\
& & & \qquad + \mu_{0110} P_{0110}
\end{array} \tag{8.135}$$

$$X_{\overline{3}} = \sum_{S(\mathbf{b})=3} \mu_{\mathbf{b}} E_{\mathbf{b}} \qquad \longmapsto \qquad \hat{\rho}(X_{\overline{3}}) = -\mu_{0111} P_{0111}$$

and the degenerate absolute quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$ in the different grades of the two approaches,

$$\mathcal{Q}^{1+}_{\hat{\pi}}: \quad \mathbf{0} = X^{+}_{\overline{1}} \vee \hat{\pi}(X^{+}_{\overline{1}}) = \lambda_{1000} X^{+}_{\overline{1}} \vee E_{1000} = \lambda^{2}_{1000} \mathbf{Z}^{+} \tag{8.136}$$

$$\mathcal{Q}^{2+}_{\hat{\pi}}: \quad \mathbf{0} = X^{+}_{\overline{2}} \vee \hat{\pi}(X^{+}_{\overline{2}}) \qquad \forall\, X^{+}_{\overline{2}} \in \Lambda^{2+}_{4} \tag{8.137}$$

$$\mathcal{Q}^{3+}_{\hat{\pi}}: \quad \mathbf{0} = X^{+}_{\overline{3}} \vee \hat{\pi}(X^{+}_{\overline{3}}) \qquad \forall\, X^{+}_{\overline{3}} \in \Lambda^{3+}_{4} \tag{8.138}$$

$$\begin{aligned} \mathcal{Q}^{1-}_{\hat{\rho}}: \quad \mathbf{0} &= X^{-}_{\overline{1}} \wedge \hat{\rho}(X^{-}_{\overline{1}}) \qquad (8.139) \\ &= -X^{-}_{\overline{1}} \wedge (\mu_{0001} P_{0001} + \mu_{0010} P_{0010} + \mu_{0100} P_{0100}) \\ &= (\mu^{2}_{0001} + \mu^{2}_{0010} + \mu^{2}_{0100}) \mathbf{Z}^{-} \\ \mathcal{Q}^{2-}_{\hat{\rho}}: \quad \mathbf{0} &= X^{-}_{\overline{2}} \wedge \hat{\rho}(X^{-}_{\overline{2}}) \qquad (8.140) \\ &= X^{-}_{\overline{2}} \wedge (\mu_{0011} P_{0011} + \mu_{0101} P_{0101} + \mu_{0110} P_{0110}) \\ &= (\mu^{2}_{0011} + \mu^{2}_{0101} + \mu^{2}_{0110}) \mathbf{Z}^{-} \\ \mathcal{Q}^{3-}_{\hat{\rho}}: \quad \mathbf{0} &= X^{-}_{\overline{3}} \wedge \hat{\rho}(X^{-}_{\overline{3}}) = -X^{-}_{\overline{3}} \wedge \mu_{0111} P_{0111} = \mu^{2}_{0111} \mathbf{Z}^{-}. \qquad (8.141) \end{aligned}$$

With respect to the differences between the signatures S^{+}, S^{-} and the elements of the matrices $\hat{B}$, $\hat{\Gamma}$ compare Corollary 7.13, especially its equations (7.69) to (7.74).

According to equation (8.134), the degenerate polarity $\hat{\pi}$ maps all the points $X^{+}_{\overline{1}}$ *not* lying in the plane $E_{1000} = -P_{0111}$ onto the plane $E_{1000} = -P_{0111}$, and the points $X^{+}_{\overline{1}}$ lying in the plane $E_{1000} = -P_{0111}$ to $\mathbf{0}$; it maps all linear complexes (including the lines) $X^{+}_{\overline{2}}$ to $\mathbf{0}$; and it maps all planes $X^{+}_{\overline{3}}$ to $\mathbf{0}$. It clearly is a highly degenerate polarity mapping.

According to equation (8.135), the degenerate polarity $\hat{\rho}$ projects all the planes $X^{-}_{\overline{1}}$ onto the points of the field of points in the plane $E_{1000} = -P_{0111}$; it projects all linear complexes (including the lines) $X^{-}_{\overline{2}}$ onto the lines of the field of lines in the plane $E_{1000} = -P_{0111}$; and it maps all points $X^{-}_{\overline{3}}$ *not* lying in the plane $E_{1000} = -P_{0111}$ onto the plane $E_{1000} = -P_{0111}$, and the points $X^{-}_{\overline{3}}$ lying in the plane $E_{1000} = -P_{0111}$ to $\mathbf{0}$.

Since the two polarities are degenerate, they satisfy $\hat{\pi}\hat{\rho}(X) = \hat{\rho}\hat{\pi}(X) = \mathbf{0}$ for all $X \in \Lambda_{n}$. Compare also Definition 7.21. Just the mappings from the points to the planes are the same for both polarities $\hat{\pi}$ and $\hat{\rho}$, i. e. they satisfy equation (4.278) from Theorem 4.39,

$$\hat{\gamma}_{0111\,0111} = \alpha_{0111\,\overline{0111}} \cdot \hat{\beta}_{\overline{0111}\,\overline{0111}} \cdot \alpha_{\overline{0111}\,0111} = 1 \cdot 1 \cdot (-1) = -1, \tag{8.142}$$

$$\hat{\gamma}_{1011\,1011} = \alpha_{1011\,\overline{1011}} \cdot \hat{\beta}_{\overline{1011}\,\overline{1011}} \cdot \alpha_{\overline{1011}\,1011} = (-1) \cdot 0 \cdot 1 = 0, \tag{8.143}$$

$$\hat{\gamma}_{1101\,1101} = \alpha_{1101\,\overline{1101}} \cdot \hat{\beta}_{\overline{1101}\,\overline{1101}} \cdot \alpha_{\overline{1101}\,1101} = 1 \cdot 0 \cdot (-1) = 0, \tag{8.144}$$

$$\hat{\gamma}_{1110\,1110} = \alpha_{1110\,\overline{1110}} \cdot \hat{\beta}_{\overline{1110}\,\overline{1110}} \cdot \alpha_{\overline{1110}\,1110} = (-1) \cdot 0 \cdot 1 = 0. \tag{8.145}$$

The different grades of the degenerate absolute quadric $\mathcal{Q}_{(\hat{\pi},\hat{\rho})}$ in the two approaches are shown in equations (8.136) to (8.141). In the plus approach, the degenerate point-wise absolute quadric is the field of points in the plane $E_{1000} = -P_{0111}$. For the linear complexes as well as for the planes, there are no restrictions. In a way, it is as in projective geometry, as if there was no absolute quadric present for the complexes and the planes in the plus approach.

In the minus approach, the plane-wise absolute quadric consists of all (imaginary) planes $X_{\overline{1}}^{-} = \sum_{S(\mathbf{b})=1} \mu_{\mathbf{b}} E_{\mathbf{b}}$ with $\mu_{0001}^2 + \mu_{0010}^2 + \mu_{0100}^2 = 0$, which touch the imaginary point-wise conic section $X_{\overline{1}}^{+} = -(\mu_{0001} P_{0001} + \mu_{0010} P_{0010} + \mu_{0100} P_{0100})$ with $\mu_{0001}^2 + \mu_{0010}^2 + \mu_{0100}^2 = 0$ lying in the field of the plane $E_{1000} = -P_{0111}$; the complex-wise absolute quadric consists of all (imaginary) complexes $X_{\overline{2}}^{-} = \sum_{S(\mathbf{b})=2} \mu_{\mathbf{b}} E_{\mathbf{b}}$ with $\mu_{0011}^2 + \mu_{0101}^2 + \mu_{0110}^2 = 0$, which touch the imaginary line-wise conic section $X_{\overline{2}}^{+} = \mu_{0011} P_{0011} + \mu_{0101} P_{0101} + \mu_{0110} P_{0110})$ with $\mu_{0011}^2 + \mu_{0101}^2 + \mu_{0110}^2 = 0$ lying in the field of the plane $E_{1000} = -P_{0111}$; and the point-wise absolute quadric (again) is the field of points in the plane $E_{1000} = -P_{0111}$.

The euclidean PGA in 3D by CHARLES GUNN [Gun11a, Gun11b] and by LEO DORST and STEVEN DE KENINCK [DDK24] is the degenerate, plane-based Clifford algebra with signature $(3, 0, 1)$ equivalent to the minus approach of the Clifford double algebra $\Gamma_{1,0,3;3,0,1}$. Compare equation (8.121). With respect to Definition 8.22, euclidean PGA describes euclidean geometry from the point of view of metric counterspace. With the metric Clifford double algebras the analytical description of euclidean geometry is supplemented by the point of view of metric space. In the case of 3D euclidean Clifford double algebra $\Gamma_{1,0,3;3,0,1}$ the point of view of metric space is of course highly degenerate.

As a conclusion, we can find in the homogeneous model of 3D euclidean geometry metric space $\Gamma_{1,0,3;3,0,1}^{+}$ and metric counterspace $\Gamma_{1,0,3;3,0,1}^{-}$.

The homogeneous model of 3D *dual* euclidean geometry is given by the metric Clifford double algebra $\Gamma_{3,0,1;1,0,3}$. Also there we can find metric space $\Gamma_{3,0,1;1,0,3}^{+}$ and metric counterspace $\Gamma_{3,0,1;1,0,3}^{-}$.

According to Theorem 5.9 of the major principe of duality, the homogeneous model of 3D euclidean geometry $\Gamma_{1,0,3;3,0,1}$ and the the homogeneous model of 3D dual euclidean geometry $\Gamma_{3,0,1;1,0,3}$ are *dual* to each other. They do not represent space and counterspace in the sense of Definitions 5.6 and 8.22. Rather inside of homogeneous euclidean geometry there is metric space and metric counterspace. And inside of homogeneous dual euclidean geometry there is also metric space and metric counterspace.

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Oliver Conradt
Goetheanum
Section for Mathematics and Astronomy
Rüttiweg 45
CH 4143 Dornach
Switzerland
e-mail: `mas@goetheanum.ch`