Projective Algebra Λ_n

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Introduction: Principle of Duality

Which role does the projective principle of duality play in physics?

- Oliver Conradt, *Mathematical Physics in Space and Counterspace*, PhD Thesis, University of Basel, Switzerland, 2000; 2$^{\text{nd}}$ edition, Dornach, 2008.
- Projective principle of duality: First stated by the French mathematicians JEAN VOICTOR PONCELET (1788-1867) and JOSEPH-DIAZ GERGONNE (1771-1859) in the first quarter of the 19$^{\text{th}}$ century.
- Oliver Conradt, *The Principle of Duality in Clifford Algebra and Projective Geometry*, in: Clifford Algebras and their Applications in Mathematical Physics, Volume 1, pp. 157-194, R. Abłamowicz and B. Fauser (Eds.), Birkäuser, 2000.

Introduction: Completely Dual Approach

- Geometric Clifford Algebra $\mathcal{G}_{p,q}(+, \ , \ )$
  - inner product $A_{\overline{r}} \cdot B_{\overline{s}} = \langle A_{\overline{r}} B_{\overline{s}} \rangle_{|r-s|}$
  - outer product $A_{\overline{r}} \wedge B_{\overline{s}} = \langle A_{\overline{r}} B_{\overline{s}} \rangle_{r+s}$
  - non-degenerate metric
  - $\mathcal{G}^k_{p,q}$ subspace of k-vectors
- Hestenes and Ziegler, *Projective Geometry with Clifford Algebra*, Acta Appl. Math., 1991, 23, pp. 25-63.
- Dual multivector $\boxed{\tilde{A} := AI^{-1} = A \cdot I^{-1}}$
- Dual geometric product $\boxed{A * B := (\tilde{A}\tilde{B})^{\sim}}$
- Dual Geometric Clifford Algebra $\mathcal{G}^{-}_{p,q}(+, \ , *)$
  - inner product $A^{-}_{\overline{r}} \circ B^{-}_{\overline{s}} = \langle A_{\overline{r}} * B_{\overline{s}} \rangle^{-}_{|r-s|}$
  - outer product $A^{-}_{\overline{r}} \vee B^{-}_{\overline{s}} = \langle A_{\overline{r}} * B_{\overline{s}} \rangle^{-}_{r+s}$
  - $\mathcal{G}^{k-}_{p,q}$ subspace of k-vectors

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Introduction: Completely Dual Approach

- Plus-minus-notation
- Double Algebra $\mathcal{G}_{p,q}(+, \ , \ , *)$: Geometric Clifford Algebra $\mathcal{G}^+_{p,q}(+, \ , \ )$ and Dual Geometric Clifford Algebra $\mathcal{G}_{p,q}(+, \ , *)$ with
  - $\mathcal{G}^+_{p,q}(+, \ ) = \mathcal{G}^-_{p,q}(+, \ )$
  - subspaces: $\mathcal{G}^{k+}_{p,q}(+, \ ) = \mathcal{G}^{(n-k)-}_{p,q}(+, \ )$
  - two inner products $\cdot$ and $\circ$
  - two outer products $\wedge$ and $\vee$
- Projective principle of duality is implemented in a
  - double Clifford algebra
  - with non-degenerating metric

Introduction: Dirac's secret tool for research

- Dirac, *Recollections of an Exciting Era*, 1977. In Weiner, C. (ed.), *History of Twentieth Century Physics*, pp. 109-146, New York, London, Academic Press.

. . . The second thing I learned from Fraser was projective geometry. Now, that had a profound influence on me because of the mathematical beauty involved in it. There was also very great power in the methods employed. I think probably most physicists know very little about projective geometry. That I would say is a failing in their education. . . .

Now, I was always much interested in the beauty of mathematics, and this introduction to me of projective geometry stimulated me very much and provided, I would say, a lifelong interest.

You might think that projective geometry is not of much interest to a physicist, but that is not so. Physicists nowadays are concerned very largely with Minkowski space. Now, if you want to picture relationships in Minkowski space, relationships between vectors and tensors, often the very best way to do it is by using the notions of projective geometry. I was continually using these ideas of projective geometry in my research work. When you want to discover how a particular quantity transforms under a Lorentz transformation, very often the best way of handling the problem is in terms of projective geometry.

It was a most useful tool for research, but I did not mention it in my published work. I do not think I have ever mentioned projective geometry in my published work (but I am not sure about that) because I felt that most physicists were not familiar with it. When I had obtained a particular result, I translated it into an analytic form and put down the argument in terms of equations. That was an argument which any physicist would be able to understand without having had this special training.

However, for the purposes of research, when one is entering into a new field and one does not know what lies in front of one, one wants very much to visualize the things which one is dealing with, and projective geometry does provide the best tool for this.

That applied also to my later work on spinors. One had quite a new kind of quantity to deal with, but for discussing the relationships between spinors, again, the ideas of projective geometry are very useful. . . .

Problem: Nature of Projective Geometry

- Projective Geometry
  - incidence relations
  - operations of connection and intersection
  - principle of duality
  - no metric
- Literature
  - Klein, *Vorlesungen über nicht-euklidische Geometrie*, Springer, 1968.
  - Locher, *Projektive Geometrie*, Dornach, 1980
  - Stoß, *Einführung in die Synthetische Liniengeometrie*, Dornach, 1999
  - Kowol, *Projektive Geometrie und Cayley-Klein Geometrien der Ebene*, Birkhäuser, 2009
  - Ziegler, *Projective Geometry and Line Geometry*, Dornach, 2012

Problem: Research Task

- Determine the double algebra corresponding to projective geometry.
- No metric!

- My method was to use synthetic projective geometry to determine projective algebra Λ_n.

- Course of action:
  1. definition of projective algebra Λ_n
  2. transition from projective to geometric Clifford algebra $\mathcal{G}_n$
  3. axioms of projective geometry thereby using projective algebra

Projective Algebra: Complementary Grading

- For a vector space V with finite grading

$$V = \bigoplus_{k=0}^{n} V_k^+$$

there is always a complementary grading

$$V = \bigoplus_{k=0}^{n} V_k^-$$

with $V_k^+ = V_{n-k}^-$.

- Plus-minus-notation, plus and minus approach

Projective Algebra: Product Signs

- We note the two (outer) products of projective algebra Λ_n by the signs $\wedge$ and $\vee$.
- Multiple outer products

$$\bigwedge_{l=1}^{m} X_l := X_1 \wedge X_2 \wedge \cdots \wedge X_m$$

$$\bigvee_{l=1}^{m} X_l := X_1 \vee X_2 \vee \cdots \vee X_m$$

Projective Algebra: Definition I

A *projective* $\mathbb{F}$*-algebra* Λ_n oder shorter *projective algebra* is a set Λ_n with four operations.

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{+} & \Lambda_n \\ (A, B) & \longmapsto & A + B \end{array} \qquad \begin{array}{ccc} \mathbb{F} \times \Lambda_n & \xrightarrow{\cdot} & \Lambda_n \\ (\alpha, A) & \longmapsto & \alpha \cdot A \end{array}$$

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{\wedge} & \Lambda_n \\ (A, B) & \longmapsto & A \wedge B \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{\vee} & \Lambda_n \\ (A, B) & \longmapsto & A \vee B \end{array}$$

The operations are called addition $(+)$, scalar multiplication (no sign or $\cdot$), major outer product $(\wedge)$ und minor outer product $(\vee)$. The obey the following conditions:

(A1) $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$.

Projective Algebra: Definition II

(A2) $\Lambda_n(+,\cdot)$ is a $\mathbb{F}$-vector space of dimension 2^n, with a complementary grading

$$\Lambda_n(+,\cdot) = \bigoplus_{k=0}^{n} \Lambda_n^{k+}(+,\cdot) = \bigoplus_{k=0}^{n} \Lambda_n^{k-}(+,\cdot), \qquad k, n \in \mathbb{N},$$

and with the following dimensions for the subspaces,

$$\dim\left(\Lambda_n^k(+,\cdot)\right) = \binom{n}{k}, \qquad 0 \leq k \leq n.$$

Projective Algebra: Definition III

(A3) $\Lambda_n(+,\cdot,\wedge)$ and $\Lambda_n(+,\cdot,\vee)$ are two associative $\mathbb{F}$-algebras without identity element. In addition the outer products live up to the requirements:

- All scalars $X_{\overline{0}}$ are left and right zero divisors.
- Outer products between homogeneous multivectors add the grades.

$$A^+_{\overline{r}} \wedge B^+_{\overline{s}} = \langle A^+_{\overline{r}} \wedge B^+_{\overline{s}} \rangle^+_{r+s} \quad r+s \leq n$$
$$A^-_{\overline{r}} \vee B^-_{\overline{s}} = \langle A^-_{\overline{r}} \vee B^-_{\overline{s}} \rangle^-_{r+s} \quad r+s \leq n$$

Projective Algebra: Definition IV

- For 1-vektors $A_i \in \Lambda_n^{1+}$ or $B_i \in \Lambda_n^{1-}$ we have with $l > 1$

$$\bigwedge_{i=1}^{l} A_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} A_1,\ A_2,\ \ldots,\ A_l \text{ are} \\ \text{linearly independent.} \end{array} \right.$$

$$\bigvee_{i=1}^{l} B_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} B_1,\ B_2,\ \ldots,\ B_l \text{ are} \\ \text{linearly independent.} \end{array} \right.$$

Projective Algebra: Combined Outer Products

Definition (Combined outer product)

Any mathematical term which contains the combined outer product $\Diamond$ can be read twice: Firstly with respect to the plus approach as major outer product $\wedge$ and seconldy with respect to the minus approach as minor outer product $\vee$.

Example: The expression

$$X_{\bar{1}} \Diamond Y_{\bar{1}} = -Y_{\bar{1}} \Diamond X_{\bar{1}} \qquad \forall\ X_{\bar{1}}, Y_{\bar{1}} \in \Lambda_n^1$$

means

$$X_{\bar{1}}^{+} \wedge Y_{\bar{1}}^{+} = -Y_{\bar{1}}^{+} \wedge X_{\bar{1}}^{+} \qquad \forall\ X_{\bar{1}}^{+}, Y_{\bar{1}}^{+} \in \Lambda_n^{1+}$$
$$X_{\bar{1}}^{-} \vee Y_{\bar{1}}^{-} = -Y_{\bar{1}}^{-} \vee X_{\bar{1}}^{-} \qquad \forall\ X_{\bar{1}}^{-}, Y_{\bar{1}}^{-} \in \Lambda_n^{1-}$$

Projective Algebra against Graßmann algebra

- A Graßmann algebra $\bigwedge V$ of a vector space V with $\dim V = n$ is a associative, unital, graded and antisymmetric algebra of dimension 2^n.
- Projective algebra Λ_n is a double algebra, this is why we can only check whether the plus approach $\Lambda_n(+, \cdot, \wedge)$ or the minus approach $\Lambda_n(+, \cdot, \vee)$ is a Graßmann algebra.
  - same properties: associative, graded, dimensions of the algebra and its subspaces, antisymmetric
  - different properties: no identity element, scalars are zero divisors

Projective Algebra: Binary Indices

- n-digit binary numbers:

$$\mathbf{b} \equiv b_{n-1} \ldots b_1 b_0 = \left[\sum_{k=0}^{n-1} b_k 2^k \right]_{10} , \qquad b_k \in \{0, 1\}$$

- Sum of digits: $S(\mathbf{b}) := \left[\sum_{k=0}^{n-1} b_k \right]_{10}$
- Binary complement: $\overline{\mathbf{b}} = \overline{b_{n-1} \ldots b_1 b_0} := \overline{b_{n-1}} \ldots \overline{b_1}\, \overline{b_0}$ with $\overline{0} := 1$ and $\overline{1} := 0$.
- lower left indices ${}_{(l_1 l_2 \ldots l_m)}\mathbf{b}$ and upper left indices ${}^{(l_1 l_2 \ldots l_m)}\mathbf{b}$

Projective Algebra: Left Indices

Example: $\mathbf{b} = [01011]_2$

Definition $^{(l_1 l_2 \ldots l_m)}\mathbf{b} := \overline{{}_{(l_1 l_2 \ldots l_m)}\mathbf{b}}$

$$S(\mathbf{b}) = 3 \qquad S(\overline{\mathbf{b}}) = 2$$

$\mathbf{b} = 01011$		$\overline{\mathbf{b}} = 10100$
${}_1\mathbf{b} = 00001$	${}^1\mathbf{b} = 11110$	${}_1\overline{\mathbf{b}} = 00100$
${}_2\mathbf{b} = 00010$	${}^2\mathbf{b} = 11101$	${}_2\overline{\mathbf{b}} = 10000$
${}_3\mathbf{b} = 01000$	${}^3\mathbf{b} = 10111$	${}_{(12)}\overline{\mathbf{b}} = 10100 = \overline{\mathbf{b}}$
${}_{(12)}\mathbf{b} = 00011$	${}^{(12)}\mathbf{b} = 11100$	
${}_{(13)}\mathbf{b} = 01001$	${}^{(13)}\mathbf{b} = 10110$	${}^1\overline{\mathbf{b}} = 11011$
${}_{(23)}\mathbf{b} = 01010$	${}^{(23)}\mathbf{b} = 10101$	${}^2\overline{\mathbf{b}} = 01111$
${}_{(123)}\mathbf{b} = 01011 = \mathbf{b}$	${}^{(123)}\mathbf{b} = 10100 = \overline{\mathbf{b}}$	${}^{(12)}\overline{\mathbf{b}} = 01011 = \mathbf{b}$

Projective Algebra: Binary Indices for m Elements

Theorem

Let V be a vector space, $A_i \in V$, $i \in \{1, \ldots, m\}$ and $\mathbf{b}$ a binary variable with m digits. Then we can label the m elements A_i with

$$V \ni A_{\mathbf{b}}, \qquad S(\mathbf{b}) = 1.$$

Proof.

With $\mathbf{b}_i = \left[2^{i-1}\right]_{10}$ we get $S(\mathbf{b}_i) = 1$ and

$$A_i = A_{\mathbf{b}_i}, \qquad 1 \leq i \leq m.$$

□

Projective Algebra: Basis

Definition (Basis of projective algebra in the plus approach)

Let $\mathbf{b}$ be a binary variable with n digits, $\{P_\mathbf{b}\}$ with $S(\mathbf{b}) = 1$ a set of n basis 1-vectors from Λ_n^{1+} and $1^+ \in \Lambda_n^{0+} \setminus \{\mathbf{0}\}$ a vector of grade 0. Then the homogeneous multivectors

$$P_\mathbf{b} = \begin{cases} 1^+, & S(\mathbf{b}) = 0, \\ \bigwedge_{l=1}^{S(\mathbf{b})} P_{{}_l\mathbf{b}}, & 0 < S(\mathbf{b}) \leq n, \end{cases}$$

form a basis for the 2^n-dimensional vector space of projective algebra Λ_n.

Projective Algebra: Basis

Definition (Basis of projective algebra in the minus approach)

Let $\mathbf{b}$ be a binary variable with n digits, $\{E_{\mathbf{b}}\}$ with $S(\mathbf{b}) = 1$ a set of n basis 1-vectors from Λ_n^{1-} and $1^- \in \Lambda_n^{0-} \setminus \{\mathbf{0}\}$ a vector of grade 0. Then the homogeneous multivectors

$$E_{\mathbf{b}} = \begin{cases} 1^-, & S(\mathbf{b}) = 0, \\ \bigvee_{l=1}^{S(\mathbf{b})} E_{{}_l\mathbf{b}}, & 0 < S(\mathbf{b}) \leq n, \end{cases}$$

form a basis for the 2^n-dimensional vector space of projective algebra Λ_n.

Notation for the basis, if the expression is true in both approaches: $\{B_{\mathbf{b}}\}$, i. e. $\{B_{\mathbf{b}}^+\} = \{P_{\mathbf{b}}\}$ and $\{B_{\mathbf{b}}^-\} = \{E_{\mathbf{b}}\}$.

Projective Algebra: Transformation of the Basis

For the basis transformations

$$E_{\mathbf{b}} = \sum_{S(\mathbf{c})=n-k} \zeta_{\mathbf{bc}} P_{\mathbf{c}}, \qquad 0 \leq S(\mathbf{b}) = k \leq n,$$

$$P_{\mathbf{b}} = \sum_{S(\mathbf{c})=n-k} \zeta_{\mathbf{bc}}^{-1} E_{\mathbf{c}},$$

we get with $S(\mathbf{b}) = S(\mathbf{d}) = k$

$$\sum_{S(\mathbf{c})=n-k} \zeta_{\mathbf{bc}} \zeta_{\mathbf{cd}}^{-1} = \sum_{S(\mathbf{c})=n-k} \zeta_{\mathbf{bc}}^{-1} \zeta_{\mathbf{cd}} = \delta_{\mathbf{bd}}$$

where $\delta_{\mathbf{bd}}$ is the Kronecker-delta-symbol,

$$\delta_{\mathbf{bd}} := \begin{cases} 1, & \mathbf{b} = \mathbf{d}, \\ 0, & \mathbf{b} \neq \mathbf{d}. \end{cases}$$

Projective Algebra: Coefficients $\alpha_{\mathbf{bc}}$

Theorem

Let $\mathbf{b}, \mathbf{c}, \mathbf{d}$ *and* $\mathbf{e}$ *be binary n-digit numbers with* $\mathbf{d} = \mathbf{b} \text{ AND } \mathbf{c}$ *and* $\mathbf{e} = \mathbf{b} \text{ XOR } \mathbf{c}$*. Then we have*

$$B_{\mathbf{b}} \lozenge B_{\mathbf{c}} = \begin{cases} \alpha_{\mathbf{bc}} B_{\mathbf{e}} & S(\mathbf{d}) = 0 \\ \mathbf{0} & S(\mathbf{d}) \neq 0 \end{cases}$$

where

$$\alpha_{\mathbf{bc}} = (-1)^{\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} c_m} = (-1)^{\sum_{l=0}^{n-2} c_l \sum_{m=l+1}^{n-1} b_m}.$$

$$\alpha_{\mathbf{bc}}\alpha_{\mathbf{cb}} = (-1)^{S(\mathbf{b})S(\mathbf{c})} \quad \text{for} \quad S(\mathbf{b} \text{ AND } \mathbf{c}) = 0,$$

$$X_{\overline{r}} \lozenge Y_{\overline{s}} = (-1)^{rs} \cdot Y_{\overline{s}} \lozenge X_{\overline{r}}.$$

Projective Algebra: Harmonic Model

The harmonic model of the projective Algebra Λ_n is given by

$$\begin{aligned} E_{\mathbf{b}} &= \alpha_{\mathbf{b}\overline{\mathbf{b}}} P_{\overline{\mathbf{b}}}, \qquad 0 \le S(\mathbf{b}) \le n, \\ P_{\mathbf{b}} &= \alpha_{\overline{\mathbf{b}}\mathbf{b}} E_{\overline{\mathbf{b}}}, \end{aligned}$$

with

$$\begin{aligned} \alpha_{\mathbf{b}\overline{\mathbf{b}}} &= (-1)^{\sum_{l=0}^{n-2} \overline{b}_l \sum_{m=l+1}^{n-1} b_m} = (-1)^{\sum_{l=1}^{n-1} b_l \sum_{m=0}^{l-1} \overline{b}_m}, \\ \alpha_{\overline{\mathbf{b}}\mathbf{b}} &= (-1)^{k(n-k)} \alpha_{\mathbf{b}\overline{\mathbf{b}}}. \end{aligned}$$

We then get with $X_{\overline{k}}^{+} = \sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} P_{\mathbf{b}}$, $Y_{\overline{k}}^{-} = \sum_{S(\mathbf{b})=k} \nu_{\mathbf{b}} E_{\mathbf{b}}$ and $0 < k < n$

$$X_{\overline{k}}^{+} \wedge Y_{\overline{k}}^{-} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] I^{+}, \quad X_{\overline{k}}^{+} \vee Y_{\overline{k}}^{-} = \left[\sum_{S(\mathbf{b})=k} \mu_{\mathbf{b}} \nu_{\mathbf{b}} \right] I^{-}.$$

Projective Algebra: Transition to Double GCA

Define two symmetric bilinear forms, one in the plus, the other in the minus approach of projective algebra Λ_n.

$$\begin{array}{rccc} B: & \Lambda_n^1 \otimes \Lambda_n^1 & \longrightarrow & \Lambda_n^0 \\ & (\ X\ ,\ \ Y\ ) & \longmapsto & B(X,Y) = B(Y,X) \end{array}$$

Projective Algebra: Transition to Double GCA

Definition (Geometric products)

Let Λ_n be a projective $\mathbb{F}$-algebra and B^+/B^- a symmetric bilinear form in the plus/minus approach. We call the operations

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \longrightarrow & \Lambda_n \\ (A, B) & \longmapsto & AB \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \stackrel{*}{\longrightarrow} & \Lambda_n \\ (A, B) & \longmapsto & A * B \end{array}$$

major geometric product (no sign) and *minor geometric product* ($*$). They obey the following conditions:

- The geometric products are associative and distributive.
- For all scalars $X_{\overline{0}}$ the geometric product is the same as the scalar multiplication.
- Contraction rule for all 1-vectors.
- For an orthogonal system $\mathcal{S} = \{X_1, X_2, \ldots, X_m\} \subset \Lambda_n^1$ the geometric and outer products are the same.

Projective Algebra: Transition to Double GCA

Contraction rule:

$$X_{\bar{1}}^{+} X_{\bar{1}}^{+} = B^{+}(X_{\bar{1}}^{+}, X_{\bar{1}}^{+}) \in \Lambda_n^{0+} \qquad \forall\ X_{\bar{1}}^{+} \in \Lambda_n^{1+}(+,\cdot)$$

$$X_{\bar{1}}^{-} * X_{\bar{1}}^{-} = B^{-}(X_{\bar{1}}^{-}, X_{\bar{1}}^{-}) \in \Lambda_n^{0-} \qquad \forall\ X_{\bar{1}}^{-} \in \Lambda_n^{1-}(+,\cdot)$$

Orthogonal system $\mathcal{S} = \{X_1, X_2, \ldots, X_m\} \subset \Lambda_n^1$:

$$\prod_{l=1}^{m} X_l = \bigwedge_{l=1}^{m} X_l \iff \left\{ \begin{array}{l} \mathcal{S}^{+} \text{ is a orthogonal system with} \\ \text{respect to the bilinear form } B^{+}. \end{array} \right.$$

$$\mathop{\Huge *}_{l=1}^{m} X_l = \bigvee_{l=1}^{m} X_l \iff \left\{ \begin{array}{l} \mathcal{S}^{-} \text{ is a orthogonal system with} \\ \text{respect to the bilinear form } B^{-}. \end{array} \right.$$

Projective Algebra: Proposal to Discuss

geometric algebras

projective algebra	$\longleftrightarrow$	metric algebra
$\Lambda_n(+, \ , \wedge, \vee)$		$\mathcal{G}_n(+, \ , \ , *)$
(double Graßmann)		double Clifford
GPA		GCA

Projective Geometry: Equivalence Relation

Definition (Equivalence relation)

Two multivectors A and B of the projective $\mathbb{F}$-algebra Λ_n are called equivalent, if and only if A and B differ in the number $\xi \in \mathbb{F} \setminus \{0\}$,

$$A \simeq B \quad :\Longleftrightarrow \quad A = \xi B.$$

Projective Geometry: Axioms I

Let $\Lambda_n(+, \cdot, \wedge, \vee)$ be a projective $\mathbb{F}$-algebra. The projective geometry of dimension 2^n is determined by the following axioms:

(A1) Elements of projective geometry.

a) There are $n+1$ different types of basic elements. Each type is represented by the homogeneous multivectors $X_{\bar{k}}$ of one of the der $n+1$ different vector subspaces Λ_n^k.

b) A multivector M of the vector space $\Lambda_n(+, \ )$ represents an element, i. e. in general of each type of basic element exactly one.

$$M = \sum_{k=0}^{n} \langle M \rangle_k .$$

Projective Geometry: Axioms II

c) Equivalent multivectors represent the same geometric locus.

(A2) Incidence relation. Two elements A und B are incident if and only if the corresponding multivectors A and B meet the conditions

$$A \wedge B = 0 \quad \text{and} \quad A \vee B = 0.$$

(A3) Intersection and connection. The geometric operation of connection corresponds to the major outer product ($\wedge$), the geometric operation of intersection to the minor outer product ($\vee$).

Real, Complex or Finite Projective Geometry

Definition

Depending on the filed $\mathbb{F}$ of the projective $\mathbb{F}$-algbera Λ_n we have

$$\begin{aligned} \mathbb{F} = \mathbb{R} \quad &\rightarrow \quad \text{real projective geometry,} \\ \mathbb{F} = \mathbb{C} \quad &\rightarrow \quad \text{complex projective geometry,} \\ \text{finite } \mathbb{F} \quad &\rightarrow \quad \text{finite projective geometry.} \end{aligned}$$

Projective Geometry: Principle of Duality

Space Λ_n^+		Counterspace Λ_n^-
X_k^+	$\leftrightarrow$	X_k^-
$\wedge$	$\leftrightarrow$	$\vee$
$\vee$	$\leftrightarrow$	$\wedge$
$\left.\begin{array}{r} A \wedge B = 0 \\ \text{and} \\ A \vee B = 0 \end{array}\right\}$	$\leftrightarrow$	$\left\{\begin{array}{l} A \vee B = 0 \\ \text{and} \\ A \wedge B = 0 \end{array}\right.$

Table: Principle of duality in projective geometry of dimension 2^n.

Proj. Geometry: Primitive Geometric Forms I

Definition (k-primitive geometric forms of grade m)

Let $X_i^{\pm} = \langle X_i \rangle_k^{\pm}$ denote $m+1$ linear independent k-vectors with $1 \leq i \leq m+1 \leq \binom{n}{k}$ and $0 \leq k \leq n$. Then the k-primitive form is given by

$$X = \sum_{i=1}^{m+1} \xi_i X_i \qquad (\xi_1, \ldots, \xi_{m+1}) \in \mathbb{F} \setminus \{\mathbf{0}\}$$

Examples in Λ_4 (n=4) with

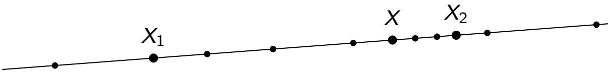
- $\Lambda_4^{0+} \rightarrow$ space of planes as a whole
- $\Lambda_4^{1+} \rightarrow$ all points of space
- $\Lambda_4^{2+} \rightarrow$ all linear complexes of space

Proj. Geometry: Primitive Geometric Forms II

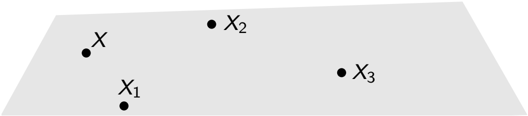
- $\Lambda_4^{3+} \to$ all planes of space
- $\Lambda_4^{4+} \to$ space of points as a whole

Primitive geometric forms:

- $k = 1,\ m = 1 \to X = \sum_{i=1}^{2} \xi_i \langle X_i \rangle_1^+$ pencil of points

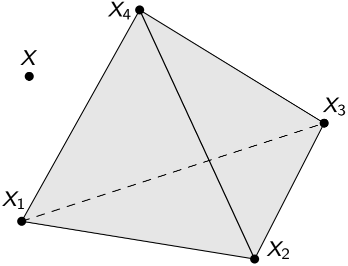


- $k = 1,\ m = 2 \to X = \sum_{i=1}^{3} \xi_i \langle X_i \rangle_1^+$ field of points



Proj. Geometry: Primitive Geometric Forms III

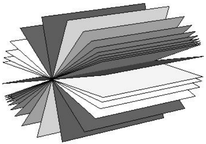
- $k = 1,\ m = 3 \rightarrow X = \sum_{i=1}^{4} \xi_i \langle X_i \rangle_1^+$ space of points



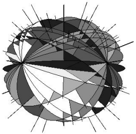
- $k = 2,\ m = 1 \rightarrow X = \sum_{i=1}^{2} \xi_i \langle X_i \rangle_2^+$ pencil of complexes
- $k = 2,\ m = 2 \rightarrow X = \sum_{i=1}^{3} \xi_i \langle X_i \rangle_2^+$ bundle of complexes
- $k = 2,\ m = 3 \rightarrow X = \sum_{i=1}^{4} \xi_i \langle X_i \rangle_2^+$ 3-manifold of compl.
- $k = 2,\ m = 4 \rightarrow X = \sum_{i=1}^{5} \xi_i \langle X_i \rangle_2^+$ 4-manifold of compl.
- $k = 2,\ m = 5 \rightarrow X = \sum_{i=1}^{6} \xi_i \langle X_i \rangle_2^+$ space of complexes

Proj. Geometry: Primitive Geometric Forms IV

- $k = 3,\ m = 1 \rightarrow X = \sum_{i=1}^{2} \xi_i \langle X_i \rangle_3^+$ pencil of planes



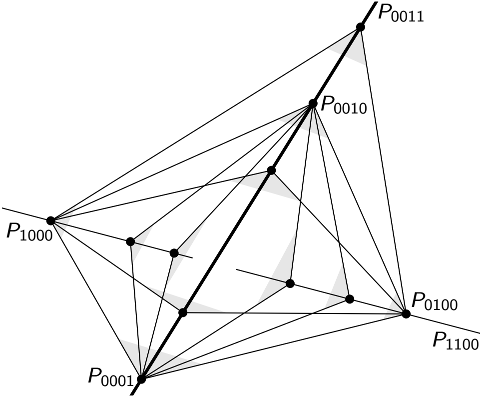
- $k = 3,\ m = 2 \rightarrow X = \sum_{i=1}^{3} \xi_i \langle X_i \rangle_3^+$ bundle of planes



- $k = 3,\ m = 3 \rightarrow X = \sum_{i=1}^{4} \xi_i \langle X_i \rangle_3^+$ space of planes

Projective Geometry: Linear Complex

Linear complex: $k = \beta_{0011} P_{0011} + \beta_{1100} P_{1100}$



$$k \wedge k = 2\beta_{0011}\beta_{1100} P_{1111}$$
$$= 0$$
$$\Leftrightarrow \beta_{0011} = 0 \text{ or } \beta_{1100} = 0$$

$$X_{\bar{1}} = \sum_{S(\mathbf{b})=1} \gamma_{\mathbf{b}} P_{\mathbf{b}} \neq \mathbf{0}$$
$$X_{\bar{1}} \wedge k \neq \mathbf{0}$$

null polarity

$$\phi(X_{\bar{1}}) := X_{\bar{1}} \wedge k$$

incident point-plane-pair

$$\phi(X_{\bar{1}}) \wedge X_{\bar{1}} = \mathbf{0}$$

Projective Geometry: Cross Ratio

Definition (Cross Ratio)

Four different basic elements

$$A = \langle A \rangle_k, \qquad B = \langle B \rangle_k, \qquad C = \langle C \rangle_k, \qquad D = \langle D \rangle_k,$$

of a k-primitive geometric form of first grade with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B$$

form the cross ratio

$$DV(AB\,CD) := \frac{\lambda}{\mu}.$$

$$T_i = \lambda_i X + \mu_i Y \rightarrow DV\left(T_1\,T_2\,T_3\,T_4\right) = \frac{\left(\frac{\lambda_1\mu_3 - \mu_1\lambda_3}{\lambda_2\mu_3 - \mu_2\lambda_3}\right)}{\left(\frac{\lambda_1\mu_4 - \mu_1\lambda_4}{\lambda_2\mu_4 - \mu_2\lambda_4}\right)}$$
$$i \in \{1, 2, 3, 4\}$$

Projective Geometry: Collineation I

A collineation is a linear mapping

$$\begin{aligned} \Phi : \quad \Lambda_n &\rightarrow \Lambda_n \\ X_{\overline{k}} &\mapsto \langle \Phi(X_{\overline{k}}) \rangle_k \end{aligned}$$

with

$$\Phi(A \wedge B) = \Phi(A) \wedge \Phi(B).$$

It is determined by $n+1$ pairs of 1-vectors (fundamental theorem of projective geometry) and—up to the factor $\det \Phi$—preserves the minor outer product too,

$$\det \phi \cdot \Phi(A \vee B) = \Phi(A) \vee \Phi(B).$$

Projective Geometry: Collineation II

Examples in Λ_4:

- Homology Φ_1: Let $Z_{\bar{1}} = P_{0001}$ be the fixed point, $Z_{\bar{3}} = P_{1110}$ the point wise fixed plane. $Z_{\bar{1}}$ and $Z_{\bar{3}}$ are not incident, i. e. $Z_{\bar{1}} \wedge Z_{\bar{3}} \neq \mathbf{0}$. And let

$$A = P_{0001} + \lambda P_{0010} \;\mapsto\; \Phi_1(A) = P_{0001} + \mu P_{0010}$$

be a point-pair related by Φ_1. Then we have

$$\begin{aligned} \Phi_1(P_{0001}) &= \lambda P_{0001} \\ \Phi_1(P_{\mathbf{b}}) &= \mu P_{\mathbf{b}} \quad \text{for } S(\mathbf{b}) = 1 \text{ and } \mathbf{b} \neq 0001 \end{aligned}$$

or

$$X_{\bar{1}} = \sum_{S(\mathbf{b})=1} \gamma_{\mathbf{b}} P_{\mathbf{b}} \;\mapsto\; \Phi_1(X_{\bar{1}}) = \gamma_{0001}(\lambda - \mu) Z_{\bar{1}} + \mu X_{\bar{1}}.$$

Projective Geometry: Collineation III

- **Elation Φ_2**: Let $Z_{\bar{1}} = P_{0001}$ be the fixed point, $Z_{\bar{3}} = P_{0111}$ the point wise fixed plane. $Z_{\bar{1}}$ and $Z_{\bar{3}}$ are incident, i. e. $Z_{\bar{1}} \wedge Z_{\bar{3}} = \mathbf{0}$. And let

$$A = P_{1000} \;\mapsto\; \Phi_2(A) = P_{0001} + \lambda P_{1000}$$

be a point-pair related by Φ_2. Then we get

$$\begin{aligned} \Phi_2(P_{\mathbf{b}}) &= \lambda P_{\mathbf{b}} \quad \text{for } S(\mathbf{b}) = 1 \text{ and } \mathbf{b} \neq 1000 \\ \Phi_2(P_{1000}) &= P_{0001} + \lambda P_{1000} \end{aligned}$$

or

$$X_{\bar{1}} = \sum_{S(\mathbf{b})=1} \gamma_{\mathbf{b}} P_{\mathbf{b}} \;\mapsto\; \Phi_2(X_{\bar{1}}) = \gamma_{1000} Z_{\bar{1}} + \lambda X_{\bar{1}}.$$

Conclusion

- Definition of projective algebra Λ_n as a double Graßmann algebra but with the scalars as zero divisors.
- Introduction of binary indices
- Transition from projective to geometric algebra by introducing two geometric products in Λ_n
- Definition of projective geometry.
- All grades represent a geometric object. In projective geometry we use the whole structure of Λ_n.
- The projective principle of duality is reflected by the structure of a double algebra where the two algebras are isomorphic.