Das harmonische Doppelverhältnis und der Schließungssatz von Poncelet. Beispiele aus der projektiven Algebra

Dr. Oliver Conradt

Goetheanum, Mathematisch-Astronomische Sektion

18. Januar, 2025

Grundlegendes & Einführung

- Geometrie / projektive Geometrie / organische Geometrie
- Raum & Gegenraum
- synthetische Methode / analytische Methode
- Anwendungen

Definition of Λ_n I

A *unity free exterior double* $\mathbb{F}$*-algebra* $\Lambda_n(+,\cdot,\wedge,\vee)$, or short *exterior double algebra*, is a set Λ_n with four operations:

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{+} & \Lambda_n \\ (A,B) & \longmapsto & A+B \end{array} \qquad \begin{array}{ccc} \mathbb{F} \times \Lambda_n & \xrightarrow{\cdot} & \Lambda_n \\ (\alpha,A) & \longmapsto & \alpha \cdot A \end{array} \tag{2.1}$$

$$\begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{\wedge} & \Lambda_n \\ (A,B) & \longmapsto & A\wedge B \end{array} \qquad \begin{array}{ccc} \Lambda_n \times \Lambda_n & \xrightarrow{\vee} & \Lambda_n \\ (A,B) & \longmapsto & A\vee B \end{array} \tag{2.2}$$

The operations are called addition $(+)$, scalar multiplication (no sign or $\cdot$), major exterior product $(\wedge)$ and minor exterior product $(\vee)$. The obey the following conditions:

P1 $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$.

Definition of Λ_n II

P2 $\Lambda_n(+,\cdot)$ is a complementary graded $\mathbb{F}$-vector space of dimension 2^n

$$\Lambda_n(+,\cdot) = \bigoplus_{k=0}^{n} \Lambda_n^{k+}(+,\cdot) = \bigoplus_{k=0}^{n} \Lambda_n^{k-}(+,\cdot), \qquad k,n \in \mathbb{N}, \tag{2.3}$$

with the dimensions

$$\dim\left(\Lambda_n^k(+,\cdot)\right) = \binom{n}{k}, \qquad 0 \le k \le n, \tag{2.4}$$

for the subspaces.

Definition of Λ_n III

P3 $\Lambda_n(+,\cdot,\wedge)$ and $\Lambda_n(+,\cdot,\vee)$ are two associative $\mathbb{F}$-algebras without identity element. In addition, both exterior products live up to the requirements:

- All scalars $X_{\overline{0}} \in \Lambda_n^0(+,\cdot)$ are left and right zero divisors,

$$X_{\overline{0}}^{+} \wedge M = M \wedge X_{\overline{0}}^{+} = \mathbf{0}, \qquad \forall\ M \in \Lambda_n(+,\cdot), \tag{2.5}$$

$$X_{\overline{0}}^{-} \vee M = M \vee X_{\overline{0}}^{-} = \mathbf{0}, \qquad \forall\ M \in \Lambda_n(+,\cdot). \tag{2.6}$$

Definition of Λ_n IV

- Exterior products between homogeneous multi vectors add the grades,

$$A_{\bar{r}}^{+} \wedge B_{\bar{s}}^{+} = \langle A_{\bar{r}}^{+} \wedge B_{\bar{s}}^{+} \rangle_{r+s}^{+}, \qquad r + s \leq n, \tag{2.7}$$

$$A_{\bar{r}}^{-} \vee B_{\bar{s}}^{-} = \langle A_{\bar{r}}^{-} \vee B_{\bar{s}}^{-} \rangle_{r+s}^{-}, \qquad r + s \leq n. \tag{2.8}$$

Definition of Λ_n V

- For 1-vectors $A_i \in \Lambda_n^{1+}$ or $B_i \in \Lambda_n^{1-}$ we have with $l > 1$

$$\bigwedge_{i=1}^{l} A_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} A_1,\ A_2,\ \ldots,\ A_l \text{ are} \\ \text{linearly dependent.} \end{array} \right. \tag{2.9}$$

$$\bigvee_{i=1}^{l} B_i = \mathbf{0} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} B_1,\ B_2,\ \ldots,\ B_l \text{ are} \\ \text{linearly dependent.} \end{array} \right. \tag{2.10}$$

Difference to Graßmann Algebras

There is no difference between the algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ and $\bigwedge V$ inasmuch as they are all associative, graded, antisymmetric and inasmuch as they have the same dimensions on the level of the whole algebra as well as on the level of their direct subspaces. The difference between the algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ and $\bigwedge V$ is that there is no identity element present in the unity free exterior algebras $\Lambda_n(+,\cdot,\wedge)$, $\Lambda_n(+,\cdot,\vee)$ — all scalars are zero divisors — and the Graßmann algebra $\bigwedge V$ is unital.

Projective Algebra Λ_n

Since projective geometry $\mathcal{P}_n$ is going to be defined in terms of the unity free exterior double $\mathbb{F}$-algebra $\Lambda_n(+, \cdot, \wedge, \vee)$ we will call the latter from now on shorter as *projective algebra* or *projective $\mathbb{F}$-algebra* $\Lambda_n(+, \cdot, \wedge, \vee)$.

Equivalence Relation and Equivalence Class

Two multi vectors A and B of a projective $\mathbb{F}$-algebra Λ_n are called *equivalent*, if and only if their homogeneous parts $\langle A\rangle_k$ and $\langle B\rangle_k$ differ each in a non zero number $\xi_k \in \mathbb{F}\setminus\{0\}$ for all k-vector parts,

$$A \simeq B \quad :\Longleftrightarrow \quad \langle A\rangle_k = \xi_k \langle B\rangle_k \quad \forall k \in \{0, 1, \ldots, n\}. \tag{4.1}$$

The corresponding equivalence class to a multi vector A is denoted by $[A]$.

Axioms for Projective Geometry $\mathcal{P}_n$ I

Let $\Lambda_n(+,\cdot,\wedge,\vee)$ be a projective $\mathbb{F}$-algebra. Projective geometry $\mathcal{P}_n$ of dimension 2^n is determined in terms of projective algebra Λ_n by the following axioms:

A1 *Elements of projective geometry.*

a. There are $n+1$ different types of *basic elements* corresponding to the $n+1$ different vector subspaces Λ_n^k of projective algebra Λ_n. The basic elements of a certain type (called *k-elements*) are represented by the homogeneous multi vectors $X_{\bar{k}}$ of one of the $n+1$ different vector subspaces Λ_n^k.

b. A multi vector M of the vector space $\Lambda_n(+,\ )$ represents an *element*, i. e. in general of each type of basic element exactly one,

$$M = \sum_{k=0}^{n} \langle M \rangle_k . \tag{4.2}$$

Axioms for Projective Geometry $\mathcal{P}_n$ II

- c Equivalent multi vectors represent the same element, i.e. all multi vectors $X \in [A]$ represent the same element as A does.

A2 *Incidence relation.* Two elements $[A]$ und $[B]$ are incident if and only if their corresponding homogeneous parts $\langle A \rangle_k$ and $\langle B \rangle_l$ meet the conditions

$$\left.\begin{array}{rcl} \langle A \rangle_k \wedge \langle B \rangle_l & = & 0, \\ \langle A \rangle_k \vee \langle B \rangle_l & = & 0, \end{array}\right\} \quad \forall\ k, l \in \{0, 1, \ldots, n\}. \tag{4.3}$$

A3 *Intersection and connection.* The geometric operation of connection corresponds to the major outer product $(\wedge)$, the geometric operation of intersection to the minor outer product $(\vee)$.

Cross ratio I

Definition (cross ratio)

Four different basic elements

$$A = \langle A \rangle_k, \qquad B = \langle B \rangle_k, \qquad C = \langle C \rangle_k, \qquad D = \langle D \rangle_k, \tag{4.4}$$

of a k-primitive geometric form with

$$\gamma C = A + \lambda B \qquad \text{and} \qquad \delta D = A + \mu B \tag{4.5}$$

form the cross ratio

$$CR(AB\,CD) := \frac{\lambda}{\mu}. \tag{4.6}$$

With respect to the cross ratio, the basic elements A and B are called *base elements*, the basic elements C and D *dividing elements*.

Cross ratio II

In order to show, that the cross ratio is well defined and does not depend on the weight factors of the basic elements A, B, C and D, we replace the latter by

$$A = \alpha' A', \qquad B = \beta' B', \qquad C = \gamma' C', \qquad D = \delta' D' \tag{4.7}$$

with $\alpha', \beta', \gamma', \delta' \in \mathbb{F} \setminus \{0\}$. Inserting the expressions of equation (4.7) into equation (4.5),

$$\gamma\gamma' C' = \alpha' A' + \lambda\beta' B', \qquad \delta\delta' D' = \alpha' A' + \mu\beta' B', \tag{4.8}$$

and dividing by α',

$$\frac{\gamma\gamma'}{\alpha'} C' = A' + \lambda\frac{\beta'}{\alpha'} B', \qquad \frac{\delta\delta'}{\alpha'} D' = A' + \mu\frac{\beta'}{\alpha'} B', \tag{4.9}$$

we get

$$CR(A'B'\,C'D') = \frac{\lambda}{\mu} = CR(AB\,CD). \tag{4.10}$$

Cross ratio III

Let

$$CR(AB\,CD) = \frac{\lambda}{\mu} =: \sigma \tag{4.11}$$

denote the cross ratio of the four different basic elements A, B, C and D according to Definition 4.1. We then have

$$CR(AB\,CD) = \sigma \qquad CR(AB\,DC) = \frac{1}{\sigma} \tag{4.12}$$

$$CR(AC\,DB) = \frac{1}{1-\sigma} \qquad CR(AC\,BD) = 1-\sigma \tag{4.13}$$

$$CR(AD\,BC) = \frac{\sigma-1}{\sigma} \qquad CR(AD\,CB) = \frac{\sigma}{\sigma-1} \tag{4.14}$$

Cross ratio IV

$$CR(BC\,DA) = \frac{\sigma}{\sigma - 1} \qquad CR(BC\,AD) = \frac{\sigma - 1}{\sigma} \tag{4.15}$$

$$CR(BD\,AC) = 1 - \sigma \qquad CR(BD\,CA) = \frac{1}{1 - \sigma} \tag{4.16}$$

$$CR(BA\,CD) = \frac{1}{\sigma} \qquad CR(BA\,CD) = \sigma \tag{4.17}$$

$$CR(CD\,AB) = \sigma \qquad CR(CD\,BA) = \frac{1}{\sigma} \tag{4.18}$$

$$CR(CA\,BD) = \frac{1}{1 - \sigma} \qquad CR(CA\,DB) = 1 - \sigma \tag{4.19}$$

$$CR(CB\,DA) = \frac{\sigma - 1}{\sigma} \qquad CR(CB\,AD) = \frac{\sigma}{\sigma - 1} \tag{4.20}$$

Cross ratio V

$$CR(DA\,BC) = \frac{\sigma}{\sigma - 1} \qquad CR(DA\,BC) = \frac{\sigma - 1}{\sigma} \tag{4.21}$$

$$CR(DB\,CA) = 1 - \sigma \qquad CR(DB\,AC) = \frac{1}{1 - \sigma} \tag{4.22}$$

$$CR(DC\,AB) = \frac{1}{\sigma} \qquad CR(DC\,BA) = \sigma \tag{4.23}$$

We will first proof the cross ratios

Cross ratio VI

$$CR(AB\,DC), \qquad CR(AC\,DB) \qquad \text{and} \qquad CR(BC\,DA). \tag{4.24}$$

$CR(AB\,DC)$: The switch of the two dividing elements follows directly from Definition 4.1,

$$CR(AB\,DC) = \frac{1}{CR(AB\,CD)}. \tag{4.25}$$

$CR(AC\,DB)$: We have to determine the dividing elements D and B in terms of the two base elements A and C. From equations (4.5) we get

$$\frac{\delta\lambda}{(\lambda-\mu)}D = A + \frac{\gamma\mu}{(\lambda-\mu)}C \quad \text{and} \quad -\lambda B = A - \gamma C. \tag{4.26}$$

Cross ratio VII

The corresponding cross ratio then is

$$CR(AC\,DB) = \frac{\gamma\mu}{(\lambda-\mu)} \cdot \left(-\frac{1}{\gamma}\right) = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\sigma} \qquad (4.27)$$
$$= \frac{1}{1-CR(AB\,CD)}.$$

$CR(BC\,DA)$: We have to determine the dividing elements D and A in terms of the two base elements B and C. From equations (4.5) we get

$$\frac{\delta}{(\mu-\lambda)}D = B + \frac{\gamma}{(\mu-\lambda)}C \quad \text{and} \quad -\frac{1}{\lambda}A = B - \frac{\gamma}{\lambda}C. \qquad (4.28)$$

Cross ratio VIII

The corresponding cross ratio then is

$$CR(BC\,DA) = \frac{\gamma}{(\mu-\lambda)} \cdot \left(-\frac{\lambda}{\gamma}\right) = \frac{\lambda}{\lambda-\mu} = \frac{\sigma}{\sigma-1} \\ = \frac{CR(AB\,CD)}{CR(AB\,CD)-1}. \tag{4.29}$$

With respect to the initial cross ratio $CR(AB\,CD)$, the three permutations of equation (4.24) generate the remaining 20 permutations.
The 24 permutations of how the cross ratio for four fixed basic elements can be formed end up in at maximum six different numbers. In case of the harmonic cross ratio $\sigma = -1$, which will be looked at into more detail in the two examples at the end of this subsection, the six values collapse into three: -1, $\frac{1}{2}$ and 2.

Cross ratio IX

The cross ratio of four different basic elements

$$T_i = \lambda_i X + \mu_i Y, \qquad i \in \{1, 2, 3, 4\}, \tag{4.30}$$

of a k-primitive geometric form is given by

$$CR\,(T_1\,T_2\,T_3\,T_4) = \frac{\left(\frac{\lambda_1\mu_3 - \mu_1\lambda_3}{\lambda_2\mu_3 - \mu_2\lambda_3}\right)}{\left(\frac{\lambda_1\mu_4 - \mu_1\lambda_4}{\lambda_2\mu_4 - \mu_2\lambda_4}\right)}. \tag{4.31}$$

Cross ratio X

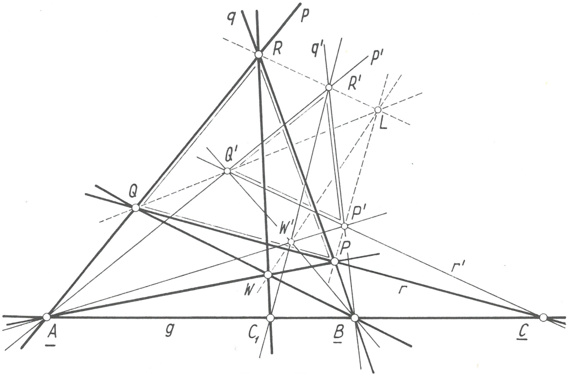


Figure: Two four-points $QRPW$ and $Q'R'P'W'$ sharing the same harmonic set of points $AB\,CC_1$. This drawing is a copy of Figure 155 from Locher, *Projektive Geometrie*.

Cross ratio XI

Let us compute the cross ratio of the harmonic point-set $AB\,CC_1$. For this we choose

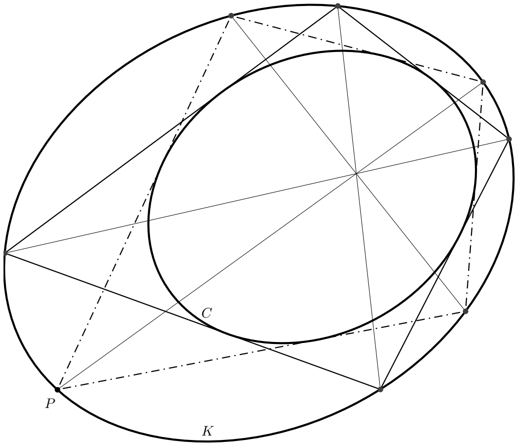
$$A := P_{001}, \qquad B := P_{010}, \qquad Q := P_{100}, \tag{4.32}$$
$$P := P_{001} + P_{010} + P_{100}.$$

Solutions:

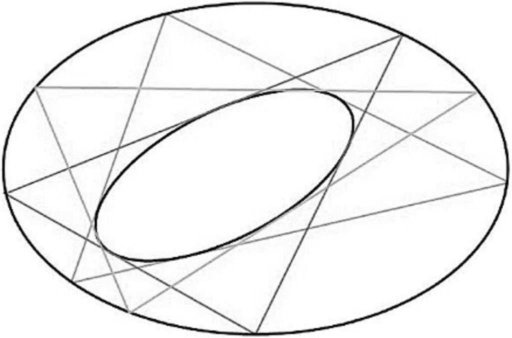
$$C \simeq A + B = P_{001} + P_{010} \tag{4.33}$$
$$W \simeq B + Q = P_{010} + P_{100}, \quad R \simeq A + Q = P_{001} + P_{100}, \tag{4.34}$$
$$C_1 \simeq A - B = P_{001} - P_{010}. \tag{4.35}$$

$$CR(AB\,CC_1) = -1. \tag{4.36}$$

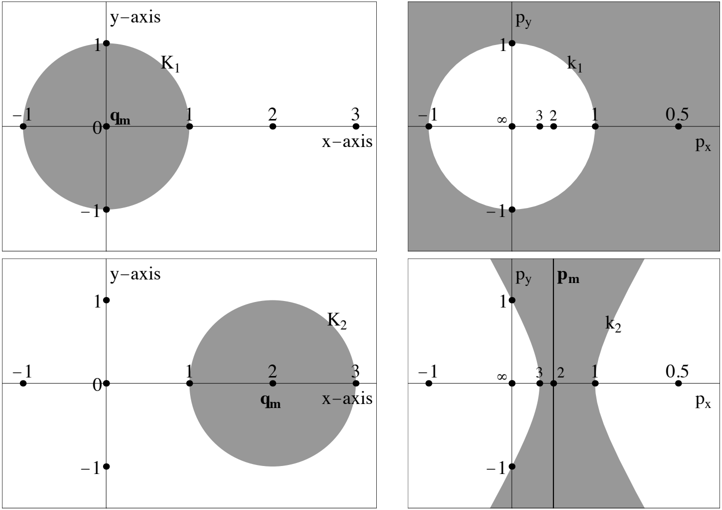
Poncelet Porism I



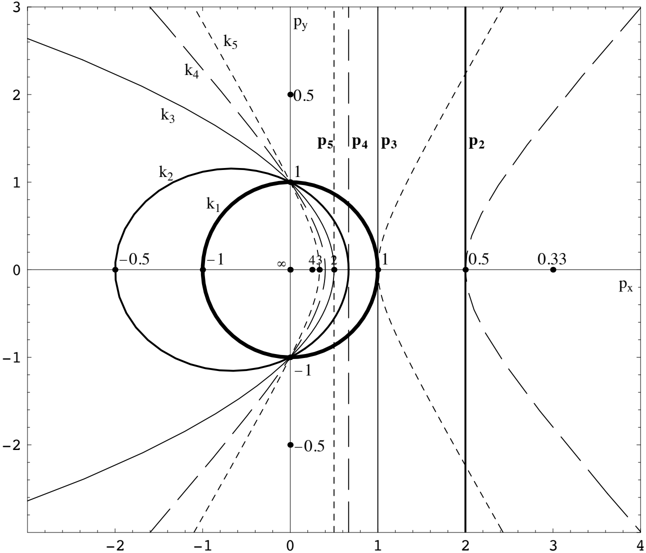
Poncelet Porism II



Poncelet Porism III



Poncelet Porism IV



Poncelet Porism V

Notation:

$$P_{0101} = \bigwedge_{l=1}^{2} P_{l0101} = P_{0001} \wedge P_{0100} \tag{4.37}$$

$$P_{1101} = \bigwedge_{l=1}^{3} P_{l1101} = P_{0001} \wedge P_{0100} \wedge P_{1000} \tag{4.38}$$

$$E_{0111} = \bigvee_{l=1}^{3} E_{l0111} = E_{0001} \vee E_{0010} \vee E_{0100} \tag{4.39}$$

$$E_{1111} = \bigvee_{l=1}^{4} E_{l1111} = E_{0001} \vee E_{0010} \vee E_{0100} \vee E_{1000}. \tag{4.40}$$

Poncelet Porism VI

Poncelet Theorem with a triangle:

$$\text{outer conic section } k' \tag{4.41}$$

$$\text{inner conic section } k \tag{4.42}$$

$$A = \alpha_{001} P_{001}, \quad B = \beta_{010} P_{010}, \quad C' = \gamma'_{100} P_{100} \tag{4.43}$$

$$C = \sum_{S(\mathbf{b})=1} \gamma_{\mathbf{b}} P_{\mathbf{b}}, \qquad \gamma_{001}\gamma_{010}\gamma_{100} \neq 0 \tag{4.44}$$

$$a = A \wedge C' = \alpha_{001}\gamma'_{100} P_{101} \tag{4.45}$$

$$b = B \wedge C' = \beta_{010}\gamma'_{100} P_{110} \tag{4.46}$$

$$c'' = A \wedge B = \alpha_{001}\beta_{010} P_{011} \tag{4.47}$$

Poncelet Porism VII

$$k^+ : \ X = \lambda^2 A + \mu^2 B + \lambda\mu\, C', \quad \alpha_{001} = \frac{(\gamma'_{100})^2}{\beta_{010}} \frac{\gamma_{001}\gamma_{010}}{(\gamma_{100})^2} \tag{4.48}$$

$$k^- : \ x = \lambda^2 a - \mu^2 b + 2\lambda\mu\, c'' \tag{4.49}$$

$$c \simeq 2P_{011} + \frac{\gamma_{100}}{\gamma_{010}} P_{101} - \frac{\gamma_{100}}{\gamma_{001}} P_{110} \tag{4.50}$$

$$A' = \alpha'_{010} P_{010} + \alpha'_{100} P_{100}, \qquad \alpha'_{010}\alpha'_{100} \neq 0 \tag{4.51}$$

$$B' = \beta'_{001} P_{001} + \beta'_{100} P_{100}, \qquad \beta'_{100}\beta'_{100} \neq 0 \tag{4.52}$$

with Brianchon $aa\, bb\, cc$

$$Z \simeq \frac{\beta'_{001}}{\beta'_{100}} P_{001} + \frac{\alpha'_{010}}{\alpha'_{100}} P_{010} + P_{100} \tag{4.53}$$

Poncelet Porism VIII

$$a'' = B \wedge C = \beta_{010}(-\gamma_{001}P_{011} + \gamma_{100}P_{110}) \quad (4.54)$$

$$b'' = C \wedge A = \alpha_{001}(-\gamma_{010}P_{011} - \gamma_{100}P_{101}) \quad (4.55)$$

$$c'' = A \wedge B = \alpha_{001}\beta_{010}P_{011} \quad (4.56)$$

with Pascal $AA\,BB\,CC$

$$P \simeq a \vee a'' \simeq \frac{\gamma_{001}}{\gamma_{100}}P_{001} + P_{100} \quad (4.57)$$

$$Q \simeq b \vee b'' \simeq \frac{\gamma_{010}}{\gamma_{100}}P_{010} + P_{100} \quad (4.58)$$

$$R \simeq c \vee c'' \simeq P_{001} - \frac{\gamma_{010}}{\gamma_{001}}P_{010}, \qquad P \wedge Q \wedge R = 0 \quad (4.59)$$

$$z = P \wedge Q$$

$$= \gamma_{001}\gamma_{010}P_{011} + \gamma_{001}\gamma_{100}P_{101} - \gamma_{010}\gamma_{100}P_{110} \quad (4.60)$$

Poncelet Porism IX

$$A' \simeq \frac{\gamma_{010}}{\gamma_{100}} P_{010} + P_{100} \tag{4.61}$$

$$B' \simeq \frac{\gamma_{001}}{\gamma_{100}} P_{001} + P_{100} \tag{4.62}$$

$$\begin{aligned} A_1 &\simeq k \vee (A \wedge A') \\ &\simeq \frac{1}{4}\gamma_{001} P_{001} + \gamma_{010} P_{010} + \frac{1}{2}\gamma_{100} P_{100} \end{aligned} \tag{4.63}$$

$$\begin{aligned} a_1 &\simeq \text{tangent of } k \text{ in } A_1 \\ &\simeq \frac{\gamma_{001}}{\gamma_{100}} P_{011} + \frac{1}{4}\frac{\gamma_{001}}{\gamma_{010}} P_{101} - P_{110} \end{aligned} \tag{4.64}$$

Poncelet Porism X

$$B_1 \simeq k \vee (B \wedge B')$$

$$\simeq 2\gamma_{001} P_{001} + \frac{1}{2}\gamma_{010} P_{010} + \frac{1}{2}\gamma_{100} P_{100} \quad (4.65)$$

$$b_1 \simeq \text{tangent of } k \text{ in } B_1$$

$$\simeq 4\frac{\gamma_{001}}{\gamma_{100}} P_{011} + 4\frac{\gamma_{001}}{\gamma_{010}} P_{101} - P_{110} \quad (4.66)$$

$$C_1 \simeq k \vee (C \wedge C')$$

$$\simeq \gamma_{001} P_{001} + \gamma_{010} P_{010} - \gamma_{100} P_{100} \quad (4.67)$$

$$c_1 \simeq \text{tangent of } k \text{ in } B_1$$

$$\simeq -2\frac{\gamma_{001}}{\gamma_{100}} P_{011} + \frac{\gamma_{001}}{\gamma_{010}} P_{101} - P_{110} \quad (4.68)$$

$$P \wedge a_1 = 0, \qquad Q \wedge b_1 = 0, \qquad Q \wedge c_1 = 0 \quad (4.69)$$

(Z, z) is with respect to k a pair of pol and polar line.

Poncelet Porism XI

$$A_1' \simeq b_1 \vee c_1 \simeq -4\gamma_{001} P_{001} + 2\gamma_{010} P_{010} + \gamma_{100} P_{100} \tag{4.70}$$

$$B_1' \simeq c_1 \vee a_1 \simeq 2\gamma_{001} P_{001} - 4\gamma_{010} P_{010} + \gamma_{100} P_{100} \tag{4.71}$$

$$C_1' \simeq a_1 \vee b_1 \simeq 4\gamma_{001} P_{001} + 4\gamma_{010} P_{010} + 5\gamma_{100} P_{100} \tag{4.72}$$

$$A_1 \wedge A \wedge A' = 0, \quad B_1 \wedge B \wedge B' = 0, \quad C_1 \wedge C \wedge C' = 0$$

$$\text{six angle } A' C_1' B' A_1' C' B_1' = k_1'$$

$$Q_1 \simeq (A' \wedge C_1') \vee (A_1' \wedge C') \tag{4.73}$$

$$\simeq 2\gamma_{001} P_{001} - 2\gamma_{010} P_{010} + \gamma_{100} P_{100} \tag{4.74}$$

$$\simeq \frac{\gamma_{001}}{\gamma_{100}} R + P, \qquad Q \simeq -\frac{\gamma_{001}}{\gamma_{100}} R + P \tag{4.75}$$

Poncelet Porism XII

$$P_1 \simeq (C_1' \wedge B') \vee (C' \wedge B_1') \tag{4.76}$$
$$\simeq -\gamma_{001} P_{001} + 2\gamma_{010} P_{010} + \gamma_{100} P_{100} \tag{4.77}$$
$$\simeq Q - \frac{\gamma_{001}}{\gamma_{100}} R, \qquad P \simeq Q + \frac{\gamma_{001}}{\gamma_{100}} R \tag{4.78}$$
$$R_1 \simeq (B' \wedge A_1') \vee (B_1' \wedge A') \tag{4.79}$$
$$\simeq \gamma_{001} P_{001} + \gamma_{010} P_{010} + 2\gamma_{100} P_{100} \tag{4.80}$$
$$\simeq P + Q, \qquad R \simeq P - Q \tag{4.81}$$