Coordinates in the theory of Klein

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This is the last article in a series of three, the previous ones being
- the handout of the conference in Holland of May 10-13, 2018, file ImTh2d20180326.pdf
- the handout of the conference in Dornach of October 19-21, 2018, file ImThKlein201810.pdf.

Parts of these will be repeated below.

Abstract A bijection is established between the projective plane as defined by Felix Klein, and the numeric complex projective plane. By proving that the ordering relation is invariant under this bijection, we show that the two planes are projectively equivalent.

Contents

1 The four planes

1.1 The geometric real plane

Let be given the real projective plane $\mathcal{S}$, with its collection of lines $\mathcal{S}_1$ and points $\mathcal{S}_0$. We add two more elements to this plane, viz. the empty set $\emptyset$ and the whole plane $\mathbb{P} = \mathbb{P}_2$. So

$$\mathcal{S} = \{\emptyset\} \cup \mathcal{S}_0 \cup \mathcal{S}_1 \cup \{\mathbb{P}\}$$

The function $\dim : \mathcal{S} \to \{-1, 0, 1, 2\}$ associates their dimension to the elements of the plane. Points have dimension 0, lines 1, the whole plane 2 and the emptyset -1. Points are denoted by upper case letters, lines by lower case ones. Also variables of $\mathcal{S}$ are denoted by lower case letters, mostly from the end of the alphabet.

The incidence realtion $\prec$ is defined by: $x \prec y$ means x lies in y, or y contains x, and this is also denoted by $y \succ x$. As usual we define the weaker relations $\preceq$ and $\succeq$. These relations are transitive. We also have

$$x \prec y \quad \Rightarrow \quad \dim(x) < \dim(y)$$

For each element $x \in \mathcal{S}$ we have $\emptyset \preceq x \prec \mathbb{P}$.

Two distinct lines, l and m, have one point in common, denoted by lm or $l \wedge m$. Two distinct points, A and B, determine one line, denoted by AB or $A \vee B$. But meet ($\wedge$) and join ($\vee$) are defined for each pair of elements of $\mathcal{S}$:

Definition 1.1 *Let be given two elements $x, y \in \mathcal{S}$. Their join $x \vee y$ is the smallest element of $\mathcal{S}$ that contains them both. Their meet $x \wedge y$ is the biggest element of $\mathcal{S}$ that is contained in each.*

Each pair of elements of $\mathcal{S}$ has exactly one join and one meet.

1.2 The numeric real projective plane

Formally the numeric real projective plane consists of two collections of homogeneous coordinates

$$\mathcal{S}'_0 = \{(x : y : z) | x, y, z \in \mathbb{R}\}, \quad \mathcal{S}'_1 = \{[x : y : z] | x, y, z \in \mathbb{R}\}$$

and as usual the empty set and the whole plane:

$$\mathcal{S}' = \{\emptyset\} \cup \mathcal{S}'_0 \cup \mathcal{S}'_1 \cup \{\mathbb{P}\}$$

In the above geometric plane $\mathcal{S}$, take any four distinct points X, Y, Z, U in general position (i.e. no three on a line). If we associate X with $(1 : 0 : 0)$, Y with $(0 : 1 : 0)$, Z with $(0 : 0 : 1)$ and U with $(1 : 1 : 1)$, each point gets a unique set of homogeneous coordinates, and so do the lines (see section 1.4). That is, for each quadruple X, Y, Z, U of points in general position we have a coordinate map $\kappa = \kappa_{XYZU}$ that associates coordinates to points and lines. We will identify $\mathcal{S}$ and $\mathcal{S}'$ as soon as there is a fixed system of reference. We will also add coordinates to the minimal and maximal element:

$$\emptyset = (0 : 0 : 0), \ \mathbb{P} = [0 : 0 : 0]$$

in any system of reference.

1.3 The Klein-plane

The Klein-plane is an extension of the geometric plane $\mathcal{S}$ as defined in section 1.1.

Definition 1.2 *An imaginary point is a projective map $P : l \to l$ with the following properties*[1]*:*
- l is a line, or rather its collection of points
- $P \neq 1_l$
- $P^3 = 1_l$
An imaginary line is a projective map $m : Q \to Q$ with the following properties:
- Q is a point, or rather the collection of lines through it
- $m \neq 1_l$
- $m^3 = 1_l$

The collection of all imaginary points (lines) is denoted by $\mathcal{T}_0$ ($\mathcal{T}_1$), and $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ is the collection of all imaginary elements. The *Klein-plane* is the set $\mathcal{U} = \mathcal{S} \cup \mathcal{T}$.

Definition 1.3 *The order relation in the Klein-plane.*

a *The imaginary point $F : l \to l$ is said to lie on its real base line l. It is on no other real line.*

b *An imaginary line $g : P \to P$ is said to go through its real base point P. It has no other real points.*

c *If P is a point* not *on line l we say that the imaginary point $F : l \to l$ is on the imaginary line $g : P \to P$ if and only if $\forall X \prec l : F(X) \vee P = g(X \vee P)$ or equivalently $\forall x \succ P : g(x) \wedge l = F(x \wedge l)$.*

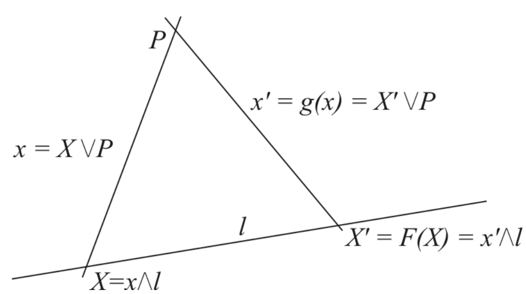

Figure 1: imaginary point on imaginary line

d *If P is on l, then for each imaginary point $F : l \to l$ and each imaginary line $g : P \to P$ holds: F is* not *on g.*

[1]You should keep in mind that there is a whole cyclic group G of automorphisms of l that represents two complex conjugated points on l. One 'half' of this group represents one imaginary point, the other half its conjugated point.

1.4 The numeric complex projective plane

A thorough introduction to numeric projective geometry is the book of Semple and Kneebone, [SempleK1952]. The numeric approach starts with a vector space, in our case the space $\mathbb{C}^3$. Numbers will be denoted by Greek letters, but coordinates will also be denoted by indexed lowercase ones: $P(p_1 : p_2 : p_3)$. And as always $i = \sqrt{-1}$.

If $P \in \mathbb{C}^3 \setminus 0$ and λ is any non-zero complex number, then P and λP represent the same point of the complex projective plane. Vectors are supposed to be vertical columns of numbers, although in running sentences we will write them horizontal to save space. In formulas we will write v^τ for the – horizontal – transpose of v. The coordinates of points will have round brackets, those of lines square ones. The symbol $\mathbb{P}_2(\mathbb{C})$ will indicate the set of points together with the set of lines. (In the literature it usually indicates only the set of points.) The real projective space $\mathbb{P}_2(\mathbb{R})$ is a subspace of it.

We take a fixed system of reference $X(1:0:0)$, $Y(0:1:0)$, $Z(0:0:1)$ and $U(1:1:1)$ for the point set. The lines $x = YZ = [1:0:0]$, $y = XZ = [0:1:0]$, $z = XY = [0:0:1]$ together with $u = [1:1:1]$ must form a system of reference for the lines, and we wish that a point $P(P_1 : P_2 : P_3)$ is on line $l[l_1 : l_2 : l_3]$ if and only if $P^\tau l = l^\tau P = P_1 l_1 + P_2 l_2 + P_3 l_3 = 0$[2]. This is established by defining u to be the line through $(1:-1:0)$, $(1:0:-1)$ and $(0:1:-1)$. As before we define:

$$\emptyset = (0:0:0),\ \mathbb{P} = [0:0:0]$$

This numeric projective plane has the comfortable property that join and meet are easy to compute with the cross product. If $x(x_1 : x_2 : x_3)$ and $y(y_1 : y_2 : y_3)$ are either distinct points or distinct lines, then the join of the points, as well as the meet of the lines is given by

$$x \times y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ -x_1 y_3 + x_3 y_1 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

This holds for complex coordinates too. If the points (lines) coincide (that is, if the vectors x, y are dependent), the cross product vanishes. A useful property of the cross product is $(x \times y)^\tau x = (x \times y)^\tau y = 0$.

An important property of projective spaces is that projectivities are regular linear maps. If T is a projective map with (point) matrix M, and that maps distinct points A and B to the distinct points $A' = MA$ and $B' = MB$ respectively, then the line AB is mapped to $A'B'$, although not by M but by line matrix $(M^\tau)^{-1}$.
Proof. Let X be an arbitrary point on an arbitrary line l, i.e. $l^\tau X = 0$. Then, using $(M^\tau)^{-1} = (M^{-1})^\tau$ and $(xy)^\tau = y^\tau x^\tau$,

$$((M^\tau)^{-1} l)^\tau (MX) = (l^\tau M^{-1})(MX) = l^\tau X = 0 \ \diamond$$

[2] $P^\tau l$ is not an inner product of vectors, but rather the mutual action of a vector P and a co-vector l. In fact it is a matrix produt of a horizontal vector P and a vertical one l. See my *Vector spaces and projective geometry*, [Boer2017].

As a consequence we have: if A and B are distinct points, and if M is a projective map, then the line AB is mapped onto[3] $(M^{-1})^\tau(A \times B) = MA \times MB$.

A warning may be appropriate. Let be given a line l with distinct points $P, Q, R = \lambda P + \mu Q$ on it, and a fourth point S not on l, see figure 2. Let $p = SP = S \times P,\ q = SQ = S \times Q,\ r = SR = S \times R$.

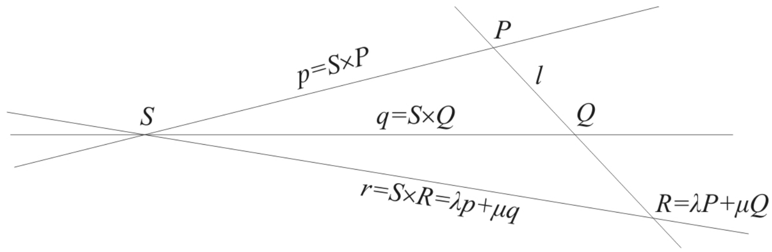

Figure 2: coordinates in a perspectivity

Then $r = S \times R = S \times (\lambda P + \mu Q) = \lambda S \times P + \mu S \times Q = \lambda p + \mu q$. So, if a range of points is perspective to a pencil of lines, then the local coordinates of a point equal the local coordinates of its perspective line. But we should be careful with applying the cross product. While $A \times B$ and $B \times A$ represent the same line, in the above formula's they have opposite signs. So if we should define $q = Q \times S$, then the local coordinates of $r = \lambda p - \mu q$ are $(\lambda : -\mu)$.

The real numerc projective plane, $\mathbb{P}_2(\mathbb{R})$, is the subspace of $\mathbb{P}_2(\mathbb{C})$ consisting of all elements of which all coordinates are real (or can be made real by multiplying by a suitable complex number).

2 Isomorphic spaces

The fundamental realation in projective geometry is that of order or incidence or containment, for which the symbols $\preceq,\ \prec,\ \succeq,\ \succ$ are used.

The operators 'meet' ($\wedge$) and 'join' ($\vee$) are defined in terms of $\preceq$:

Definition 2.1 *Let S be a projective space of dimension 2 or more[4]. The meet $a \wedge b$ of two elements $a, b \in S$ is the biggest element x of S for which holds $x \preceq a$ and $x \preceq b$. Their join $a \vee b$ is the smallest element y of S for which holds $a \preceq y$ and $b \preceq y$.*

Then we have for all $a, b, x \in S$:

$$(x \preceq a \text{ AND } x \preceq b) \Rightarrow x \preceq a \wedge b$$
$$(x \succeq a \text{ AND } x \succeq b) \Rightarrow x \succeq a \vee b$$

[3] Algebraically the formula is $|M|(M^{-1})^\tau(A \times B) = MA \times MB$.
[4] S cointains points, lines etc., and also the emptyset and the whole space.

In addition we have for all $a, b \in S$:

$$a = a \wedge b \quad \Leftrightarrow \quad a \preceq b \quad \Leftrightarrow \quad a \vee b = b$$

The rest of this section duplicates a part of [Boer2009].

Definition 2.2 *Let be given two projective spaces S, S' of dimension 2 or more. A map $f : S \to S'$ is called a* homomorphism *if for every $a, b \in S$:*

$$f(a \vee b) = f(a) \vee' f(b)$$

and

$$f(a \wedge b) = f(a) \wedge' f(b)$$

Unless confusion is likely we will omit the primes after the symbols $\preceq, \wedge$ and $\vee$.

Proposition 2.3 *Each homomorphism is order preserving.*

Proof. $a \preceq b \Rightarrow a \wedge b = a \Rightarrow f(a) \wedge f(b) = f(a \wedge b) = f(a) \Rightarrow f(a) \preceq f(b)$. $\diamond$

The converse of proposition 2.3 is not true: not every order preserving map is a homomorphism.

Proposition 2.4 *If f is a bijective homomorphism, then so is f^{-1}.* $\diamond$

Definition 2.5 *An* isomorphism *is a bijective homomorphism*[5]. *The spaces S and S' are called* isomorphic *if there exists an isomorphism between them.*

Proposition 2.6 *A map $f : S \to S'$ is an isomorphism if and only if it is bijective and for every $a, b \in S$:*

$$a \preceq b \;\Leftrightarrow\; f(a) \preceq f(b)$$

Proof. We owe this proof to Jacobson, see [Jacobson1951]. If f is an isomorphism then both f and f^{-1} are bijective homomorphisms and hence order preserving. This proves half of the statement. Conversely let f be bijective and for every $a, b \in S : a \preceq b \;\Leftrightarrow\; f(a) \preceq f(b)$. Since $a \wedge b \preceq a$ and $a \wedge b \preceq b$, also $f(a \wedge b) \preceq f(a)$ and $f(a \wedge b) \preceq f(b)$, hence $f(a \wedge b) \preceq f(a) \wedge f(b)$. Now let $x \in S'$ be a lower bound of $f(a)$ and $f(b)$, so $x \preceq f(a) \wedge f(b)$, and let $y = f^{-1}(x)$. Clearly $y \preceq a$, $y \preceq b$ and $y \preceq a \wedge b$. Then again $x \preceq f(a \wedge b)$ which means that $f(a \wedge b)$ is greatest lower bound of $f(a)$ and $f(b)$, hence equal $f(a) \wedge f(b)$. Analogous for $\vee$. $\diamond$

Proposition 2.7 *A map $f : S \to S'$ is an isomorphism if and only if it is bijective and for every $a, b \in S$: $a \prec b \;\Leftrightarrow\; f(a) \prec f(b)$.* $\diamond$

Of course the inverse of an isomorphism is again an isomorphism.

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[5] Isomorphisms are semi-linear maps. The cross ratio fails to be always invariant under isomorphisms. Therefore in the literature *projective* maps are usually defined as linear maps. These do leave the cross ratio invariant. If the ground field is $\mathbb{R}$ - or any other field with no non-trivial automorphisms - these definitions coincide, else they differ. If for instance the field is $\mathbb{C}$, then there are many non-trivial automorphisms, among them complex conjugation. So, complex conjugation is an automorphism, but not a projective map.

3 The isomorphism

We constructed a new plane, the Klein-plane, that was ment to be a projective plane over the complex numbers. Without proof we state that it is a projective plane. We will, however, establish a bijection to the numeric plane. Then we get tools to associate coordinates to the newly introduced imaginary elements, which is of great practical use.

3.1 The coordinate map κ

So we want to assign complex coordinates to the imaginary points and lines of the Klein-plane, and verify that the order relation $\prec$ is kept by that assignment. We can, however, presume an isomorphism κ between the real elements $\mathcal{S}$ and the real part $\mathbb{P}_2(\mathbb{R})$ of the numeric plane. So, in our Klein-plane we have a real system of reference $XYZU$ as described in section 1.4, and κ simply assigns the real coordinates to real points and real lines.

On the line $XY = [0 : 0 : 1]$ we define a map P_i that moves the point $X(1 : 0 : 0)$ to $I_1(-1 : \sqrt{3} : 0)$, I_1 to $I_2(1 : \sqrt{3} : 0)$ and I_2 to X. If we drop the third coordinate we find that the matrix of P_i is $(1, -\sqrt{3}; \sqrt{3}, 1)$. It has eigenvecotrs $(\pm i : 1)$ and we will associate P_i with $(i : 1)$. Then it is only natural to associate the plane coordinates $\kappa(P_i) = (i : 1 : 0)$ to P_i, and $\kappa(\overline{P}_i) = (-i : 1 : 0)$ to $\overline{P}_i$. Remember that for any non-zero *complex* number λ, also $(\lambda i : \lambda : 0)$ are homogeneous coordinates of P_i.

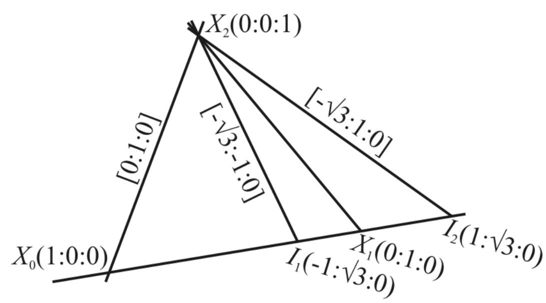

Figure 3: the fundamental imaginary line

Dually, in the point $Z(0 : 0 : 1)$ we observe the lines $ZX[0 : 1 : 0]$, $ZI_1[-\sqrt{3} : -1 : 0]$ and $ZI_2[-\sqrt{3} : 1 : 0]$ (see the warning at the end of section 1.4). Define the imaginary line l_i to be the Klein-map by $l_i(ZX) = ZI_1$ and $l_i(ZI_1) = ZI_2$. Dropping each third coordinate it has the same matrix as P_i so we define $\kappa(l_i) = [i : 1 : 0]$. Observe that $P_i \prec l_i$ and that $\kappa(l_i)^{\tau}\kappa(P_i) = 0$ indeed. Thus we fixed one imaginary line and one imaginary point on this line. They are the *fundamental* imaginary elements of the plane with respect to the system of reference $XYZU$.

Next take an arbitrary imaginary point $F : l \to l$. Let P, Q, R be distinct points of l and $F(P) = Q$, $F(Q) = R$. There are many projectivities of the plane that map X to P, I_1 to

Q and I_2 to R. Let M be any such regular *real* projective map. We would like to assign the coordinates $MP_i = M(i:1:0)^\tau$ to F. But then the question arises, if N is a second matrix that moves X to P, I_1 to Q and I_2 to R, is then NP_i a – possibly complex – multiple of MP_i? The answer is yes, and that is a consequence of the following proposition.

Proposition 3.1 *In the real projective plane let be given the line l and three distinct points P, Q, R on it. Let $F = F_{PQR}$ be the imaginary point represented by P, Q, R. Let M, N be regular real projective maps, each mapping X, I_1, I_2 to P, Q, R respectively. Then there is non-zero real number α such that $M(i:1:0)^\tau = \alpha N(i:1:0)^\tau$.*

Proof. Since P, Q, R are distinct there is a non-zero real number ρ such that $R = \rho P + Q$. Let $M = (m_{jk}) = (M_0, M_1, M_2)$ where the M_j are the column vectors of M. Then we have

$$MX = (M_0, M_1, M_2)(1:0:0)^\tau = M_0 = \lambda P, \quad \lambda \in \mathbb{R}^*$$

and

$$MI_1 = (\lambda P, M_1, M_2)(-1:\sqrt{3}:0)^\tau = -\lambda P + M_1\sqrt{3} = \mu Q, \quad \mu \in \mathbb{R}^*$$

hence $M_1 = (\lambda P + \mu Q)/\sqrt{3}$. Then

$$\begin{aligned} MI_2 &= (\lambda P, \tfrac{\lambda P + \mu Q}{\sqrt{3}}, M_2)(1:\sqrt{3}:0)^\tau = \lambda P + \lambda P + \mu Q = \\ &= 2\lambda P + \mu Q = \eta(\rho P + Q), \quad \eta \in \mathbb{R}^* \end{aligned}$$

or $2\lambda P + \mu Q = \rho\eta P + \eta Q$. Since P and Q are independent vectors, we have $2\lambda = \rho\eta$ and $\mu = \eta = 2\lambda/\rho$. Now our matrix is

$$M = (\lambda P, \frac{\lambda\rho P + 2\lambda Q}{\rho\sqrt{3}}, M_2)$$

and

$$M(i:1:0)^\tau = \lambda i P + \frac{\lambda\rho P + 2\lambda Q}{\rho\sqrt{3}} = \lambda(i\rho\sqrt{3}P + \rho P + 2Q)$$

This is independent of the M_j. In a similar way we find $N(i:1:0)^\tau = \lambda'(i\rho\sqrt{3}P + \rho P + 2Q)$. If now we take $\alpha = \lambda/\lambda'$ the proof is complete. $\diamond$

Thus, given the distinct points P, Q and $R = \rho P + Q$ on one line l we define

$$\kappa(F_{PQR}) = \rho(1 + i\sqrt{3})P + 2Q = (\rho P + 2Q) + i\rho\sqrt{3}P \tag{1}$$

For imaginary lines we have a similar procedure, with slightly different results. Let l, m and $n = \rho l + m$ be real lines and let $g = g_{lmn}$. $M = [M_0, M_1, M_2]$ is the matrix that maps line ZX onto l, ZI_1 onto m and ZI_2 onto n. Then we have

$$M(ZX) = [M_0, M_1, M_2][0:1:0]^\tau = M_1 = \lambda l$$

$$M(ZI_1) = [M_0, \lambda l, M_2][-\sqrt{3}:-1:0]^\tau = -M_0\sqrt{3} - \lambda l = \mu m$$

hence $M_0 = (-\lambda l - \mu m)/\sqrt{3}$.

$$\begin{aligned} M(ZI_2) &= [(-\lambda l - \mu m)/\sqrt{3}, \lambda l, M_2][-\sqrt{3}:1:0]^\tau = \\ &= 2\lambda l + \mu m = \nu(\rho l + m) \end{aligned}$$

This gives $\mu = \nu = 2\lambda/\rho$ and

$$\begin{aligned} M(l_i) &= [(-\lambda l - 2\lambda m/\rho)/\sqrt{3}, \lambda l, M_2][i : 1 : 0]^\tau = \\ &= -\tfrac{\lambda}{\rho\sqrt{3}}((\rho l + 2mi) + l\sqrt{3}) \end{aligned}$$

Thus we find

$$\kappa(g) = (\rho\sqrt{3})l - (\rho l + 2m)i = \rho(\sqrt{3} - i)l - 2mi \tag{2}$$

3.2 Consistency

We will now prove that, given two pairs of points on an imaginary line, each will lead to the same set of coordinates for that line.

In the real projective plane let be given the lines m and n with meeting point S and a point P neither on m nor on n, see figure 4. Let $x = PS$ and let y, y' be distinct lines through P but not through S. Define $Q = y \wedge m$, $R = y \wedge n$, $Q' = y' \wedge m$ and $R' = y' \wedge n$. Let $F = F_{SQQ'}$ and $G = G_{SRR'}$ be imaginary points and $l = l_{xyy'}$ an imaginary line.

We will prove that $\kappa(P \vee F) = \kappa(F \vee G)$.

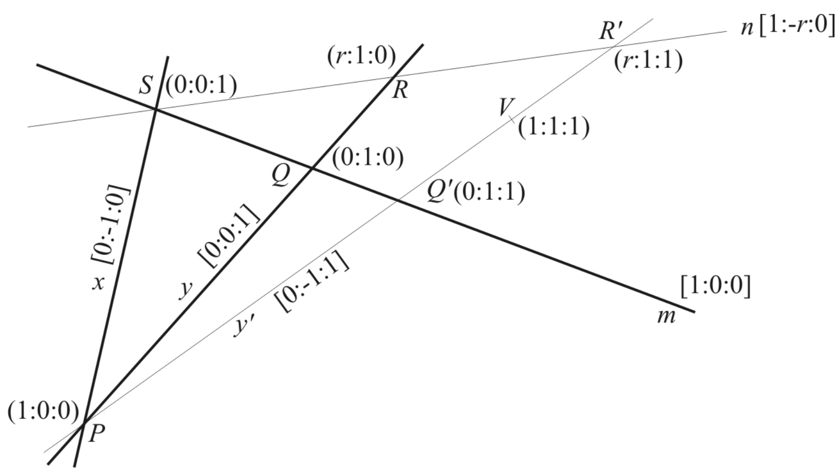

Figure 4: consistency

Let V be an arbitrary point on y' but not on any of the other four lines. Let M be the coordinate transformation that changes the system of reference from $XYZU$ to $PQSV$. Then after applying M we have the following coordinates: $P(1 : 0 : 0)$, $Q(0 : 1 : 0)$, $S(0 : 0 : 1)$, $m[1 : 0 : 0]$, $x = P \times S = [0 : -1 : 0]$, $y = P \times Q = [0 : 0 : 1]$, $y' = P \times V = [0 : 1 : -1]$. Take $R(\rho : 1 : 0)$ and verify that it is on y. Then $n = [1 : -\rho : 0]$, $R = (\rho : 1 : 1)$, $Q' = (0 : 1 : 1)$

Now we see that $Q' = S + Q$, $R' = S + R$ and $y' = x + y$. From section 3.1 we know that

$$\kappa(F_{SQQ'}) = (1+i\sqrt{3})S + 2Q = \begin{pmatrix} 0 \\ 2 \\ 1+i\sqrt{3} \end{pmatrix}$$

and

$$\kappa(G_{SRR'}) = (1+i\sqrt{3})S + 2R = \begin{pmatrix} 2\rho \\ 2 \\ 1+i\sqrt{3} \end{pmatrix}$$

so

$$P \times F_{SQQ'} = P \times G_{SRR'} = \begin{bmatrix} 0 \\ -1-i\sqrt{3} \\ 2 \end{bmatrix}$$

$$F_{SQQ'} \times G_{SRR'} = \begin{bmatrix} 0 \\ 2\rho(1+i\sqrt{3}) \\ -4\rho \end{bmatrix} = \begin{bmatrix} 0 \\ -(1+i\sqrt{3}) \\ 2 \end{bmatrix}$$

So, each pair of points of l gives the same value of $\kappa(l)$. That implies that for each (real or imaginary) point H of l we have $(\kappa(l))^\tau \kappa(H) = 0$.

If we go back to the original coordinates by applying M^{-1} to the points, and M^τ to l, we get

$$M^{-1}H \times M^{-1}H' = M^\tau(H \times H')$$

Since the right part is independent of the points H, H' on l, so is the left part.

3.3 The inverse of κ

Given an imaginary point $Z(a_0+b_0i : a_1+b_1i : a_2+b_2i) = A+iB$ with A and B real, what is the Klein-map that has these coordinates? First observe that $\overline{Z} = (a_0-b_0i : a_1-b_1i : a_2-b_2i)$ is on the same real line l as Z. But also the real and imaginary parts, $A(a_0 : a_1 : a_2)$ and $B(b_0 : b_1 : b_2)$ are on l. (Observe that by hypothesis neither one can be the zero vector! For if $B=0$ then Z is real. But if $A=0$ then $Z=iB=B$, hence real too.) So the coordinates of l are simply those of $A \times B$. Now every real point of l can be written as $\lambda A + \mu B$, with λ, μ real and not both 0. Next, take three distinct real points $P = A$, $Q = \lambda A + B$ and $R = \mu A + B$, $\lambda \neq \mu$ on l. From this follows $R = (\mu-\lambda)P + Q = \rho P + Q$, $\quad \rho = \mu - \lambda$. There exixts a map M that moves X, I_1, I_2 onto P, Q, R respectively and the coordinates of the imaginary point $F = F_{PQR}$ are by equation (1):

$$\kappa(F_{PQR}) = \rho(1+i\sqrt{3})P + 2Q = (\mu-\lambda)(1+i\sqrt{3})A + 2(\lambda A + B)$$

And this result must equal (a multiple of) the coordinates of Z:

$$(\mu-\lambda)(1+i\sqrt{3})A + 2(\lambda A + B) = (\gamma+\delta i)(A+iB), \quad \gamma,\delta \in \mathbb{R}$$

$$\Big((\mu-\lambda) + i(\mu-\lambda)\sqrt{3} + 2\lambda\Big)\, A + 2B = (\gamma+\delta i)(A+iB)$$

Since A and B are independent we must have

$$\mu + \lambda + i(\mu - \lambda)\sqrt{3} = \gamma + \delta i \quad \text{and} \quad 2 = \gamma i - \delta$$

From the last equation follows $\gamma = 0,\ \delta = -2$. Then we find $\lambda = 1/\sqrt{3},\ \mu = -1/\sqrt{3}$.

So, given an imaginary point $Z = A + iB$, its corresponding Klein-map is F_{PQR} with

$$P = A,\ Q = \frac{A}{\sqrt{3}} + B,\ R = -\frac{A}{\sqrt{3}} + B \tag{3}$$

Check that if we substitute these values of P, Q in equation (1), we get (a multiple of) $A + iB$ again.

In a similar way we find the inverse for lines. Let a, b be real lines, $l = a + ib,\ \bar{l} = a - ib$ and $S = l \wedge \bar{l} = a \times b$. Put $p = a,\ q = \lambda a + b,\ r = \mu a + b = (\mu - \lambda)p + q = \rho p + q$. By equation 2 we find

$$(\mu - \lambda)(\sqrt{3} - i)a - 2i(\lambda a + b) = (\gamma + \delta i)(a + ib)$$

This leads to $\lambda = 1/\sqrt{3},\ \mu = -1/\sqrt{3},\ \gamma = -2,\ \delta = 0$ or, surprisingly, to exactly the same formula as for points:

$$p = a,\ q = \frac{a}{\sqrt{3}} + b,\ r = -\frac{a}{\sqrt{3}} + b \tag{4}$$

Thus we proved that κ is bijective.

Summarizing we have:

Definition 3.2 *Given a fixed system of reference $XYZU$, an arbitrary line l with distinct points P, Q and $R = \rho P + Q$ on it, and an arbitrary point S with lines l, m and $n = \rho l + m$ through it. Let $F = F_{PQR}$ and $g = g_{lmn}$. Then κ is defined by*

$$\kappa(F) = \rho(1 + i\sqrt{3})P + 2Q = (\rho P + 2Q) + i\rho\sqrt{3}P$$

$$\kappa(g) = (\rho\sqrt{3})l - (\rho l + 2m)i = \rho(\sqrt{3} - i)l - 2mi$$

Proposition 3.3 *Given the real points A, B, the real lines a, b, the imaginary point $A + iB$ and the imaginary line $a + ib$, there are real collinear points*

$$P = A,\ Q = \frac{A}{\sqrt{3}} + B,\ R = -\frac{A}{\sqrt{3}} + B$$

and real concurrent lines

$$p = a,\ q = \frac{a}{\sqrt{3}} + b,\ r = -\frac{a}{\sqrt{3}} + b$$

such that $\kappa^{-1}(A + iB) = F_{PQR}$ and $\kappa^{-1}(a + ib) = g_{pqr}$. $\diamond$

Proposition 3.4 *The map κ is bijective.* $\diamond$

4 The invariance of $\prec$

Nex we have to prove that the order relation in the plane is invariant under κ and under its inverse (see section 2). So if P is a (real or imaginary) point and l a (real or imaginary) line l, we have to prove that $P \prec l \Leftrightarrow \kappa(P) \prec \kappa(l)$ or, equivalently, $P \prec l \Leftrightarrow \kappa(P)^\tau \kappa(l) = 0$. In the sequel we will omit κ and write $P^\tau l$ instead.

We distinguish the following cases.

- If P and l are both real, this is classical real projective geometry.
- If $F : m \to m$ is imaginary and l real then $F \prec l$ if and only if $l = m$. Let A be an arbitrary point of l and let $A' = F(A)$, $A'' = F(A')$. Let M be any map that moves X, I_1, I_2 onto A, A', A'' respectively, then $F = M(i : 1 : 0)^\tau$. Now $(M^{-1})^\tau$ moves line XY to m, so the coordinates of m are $(M^{-1})^\tau[0 : 0 : 1]^\tau$. If $l = m$ then

$$F^\tau l = (i : 1 : 0)M^\tau(M^{-1})^\tau[0 : 0 : 1]^\tau = (i : 1 : 0)[0 : 0 : 1]^\tau = 0$$

  Conversely, if $F^\tau l = 0$ then $(i : 1 : 0)[\alpha : \beta : \gamma] = 0$ hence $\beta = -\alpha i$ and $l = (M^{-1})^\tau[\alpha : -\alpha i : \gamma]^\tau$. But l is real, hence $\alpha = 0$ and $l = \gamma(M^{-1})^\tau[0 : 0 : 1]^\tau = m$.
- And dually a similar proof for a real point on an imaginary line.
- Now let $F : l \to l$ and $g : Q \to Q$ be imaginary elements with $Q \not\prec l$. Take an arbitrary point $A \prec l$ and define $A' = F(A)$, $A'' = F(A')$, $a = QA$, $a' = QA'$, $a'' = QA''$. Then there is an $\alpha \in \mathbb{R}^*$ such that $A'' = \alpha A + A'$. Now the map $M = (\alpha\sqrt{3}A, \alpha A + 2A', Q)$ sends X to A, I_1 to A', I_2 to A'' and Z to Q (and U to $\alpha(1 + \sqrt{3})A + 2A' + Q$). M^{-1} sends them back, M^τ sends l to XY and $F = MP_i$.
  F is on g if and only if $g(a) = a'$ and $g(a') = a''$, and that is the case if and only if $M^\tau g = l_i$, the fundamental imaginary line. In that case $F^\tau g = (MP_i)^\tau(M^{-1})^\tau l_i = P_i^\tau M^\tau(M^{-1})^\tau l_i = P_i^\tau l_i = 0$.
  Conversely, let $g = [\alpha : \beta : \gamma]$. If $F^\tau g = (MP_i)^\tau g = 0$ then $(P_i)^\tau M^\tau g = 0$ so $M^\tau g = [\alpha : -\alpha i : \gamma]$. But g contains $Q = M(0 : 0 : 1)^\tau$ so $\gamma = 0$ and $M^\tau g = -\alpha i[i : 1 : 0] = l_i$. That means $P_i \prec M^\tau g$, or $F = MP_i \prec (M^\tau)^{-1}(M^\tau g) = g$.

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