1 Computations in space

1.1 Coordinates

1.1.1 Points and planes

A system of reference in our real projective 3-space $\mathcal{S}$, consists of five points in general position. We will call and coordinatize them as follows. $X_0(1:0:0:0), X_1(0:1:0:0), X_2(0:0:1:0), X_3(0:0:0:1), X_u(1:1:1:1)$.

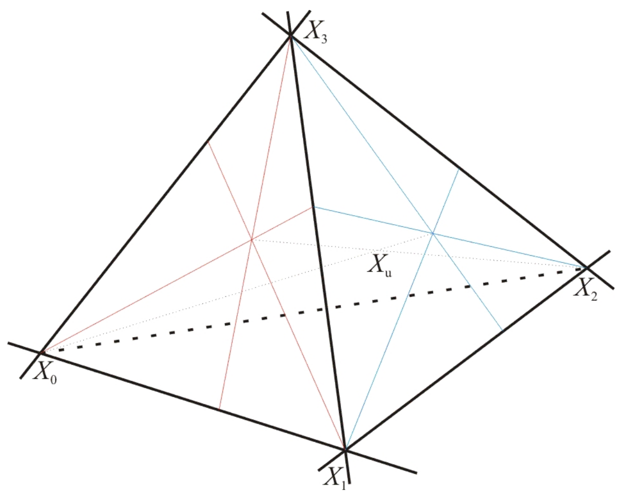

Figure 1: the system of reference

In addition we have the five coordinate planes $Y_0 = X_1X_2X_3 = [1:0:0:0]$, $Y_1 = X_0X_2X_3 = [0:1:0:0]$, $Y_2 = X_0X_1X_3 = [0:0:1:0]$, $Y_3 = X_0X_1X_2 = [0:0:0:1]$, and $Y_u = [1:1:1:1]$ which is the plane through the points $(1:-1:0:0)$, $(1:0:-1:0)$, $(1:0:0:-1)$, $(0:1:-1:0)$, $(0:1:0:-1)$ and $(0:0:1:-1)$. A point $P(p_i)$ is in a plane $A[a_j]$ if and only if

$$A^\tau P = [a_0:a_1:a_2:a_3]\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3 = 0$$

This is not an inproduct but a matrix multiplication.

If $P(p_i), Q(q_i)$ are points then for each pair of real numbers λ, μ the point $R = \lambda P + \mu Q = (\lambda p_i + \mu q_i)$ is on the line PQ. Dually the plane $C = \lambda A + \mu B$ contains the common line of the planes A and B.

Sometimes we need coordinates too for the minimal and maximal element:

$$\emptyset = (0:0:0:0), \quad \mathbb{P} = [0:0:0:0]$$

1.1.2 Plücker-coordinates

Let $P(p_i)$ and $Q(q_i)$ be distinct points. The *Plücker-matrix* of these points is the 4×4-matrix $Pl_m(l) = (l_{ij})$ with

$$l_{ij} = p_i q_j - p_j q_i \tag{1}$$

If $P'(p'_i), Q'(q'_i)$ are distinct points on the line PQ, and if $l'_{ij} = p'_i q'_j - p'_j q'_i$, then there is a non-zero real number λ such that $l'_{ij} = \lambda l_{ij}$ for all i, j. So we can call (l_{ij}) *the* Plücker-matrix of line $l = P \vee Q$, notation $Pl_m(l)$. It is skew symmetric and singular.

Since $l_{ij} = -l_{ji}$, and in particular $l_{ii} = 0$, there are only six significant numbers in this matrix. Depending on the author and on the particular application, in the literature there are several ways of selecting and ordering these six numbers. We define the *Plücker-vector* of l to be

$$Pl_v(l) = P \wedge_e Q = (l_{01} : l_{02} : l_{03} : l_{12} : l_{\mathbf{31}} : l_{23})$$

(Be aware of the strange fifth coordinate $l_{31} = -l_{13}$!) Here $\wedge_e$ is the *exterior* or *outer* or *wedge* product as defined by (1). These six numbers are not independent. They satisfy the so called *line condition*:

$$Pl_n(l) = l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0 \tag{2}$$

A vector $(l_{01} : l_{02} : l_{03} : l_{12} : l_{31} : l_{23})$ represents a line if and only if relation (2) holds.

The four vectors $(l_{i0} : l_{i1} : l_{i2} : l_{i3})$ are – if they are not the zero-vector – points on l, viz. its meeting points with the coordinate planes.

Dually, given the distinct planes $A(a_i)$ and $B(b_i)$ containing l we get the dual Plücker-coordinates $m_{ij} = a_i b_j - a_j b_i$. It appears that

$$m_{01} : m_{02} : m_{03} : m_{12} : m_{31} : m_{23} = l_{23} : l_{31} : l_{12} : l_{03} : l_{02} : l_{01}$$

The four vectors $[m_{i0} : m_{i1} : m_{i2} : m_{i3}]$ are, if they are not the zero-vector, planes through l, viz. the joins of l and the coordinate points.

The matrix

$$\iota = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

transforms point- into plane-coordinates and vice versa. If $(a_0 : a_1 : a_2 : a_3 : a_4 : a_5)$ is the pointwise vector of a line, then

$$\iota(a_0 : a_1 : a_2 : a_3 : a_4 : a_5) = (a_0 : a_1 : a_2 : a_3 : a_4 : a_5)^\iota = [a_5 : a_4 : a_3 : a_2 : a_1 : a_0]$$

is its planewise vector.

Observe that the relation between pointwise and planewise (=dual) matrices is a bit more complicated. We use ι also as a map defined on matrices:

$$\iota\begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ -a_0 & 0 & a_3 & -a_4 \\ -a_1 & -a_3 & 0 & a_5 \\ -a_2 & a_4 & -a_5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ -a_0 & 0 & a_3 & -a_4 \\ -a_1 & -a_3 & 0 & a_5 \\ -a_2 & a_4 & -a_5 & 0 \end{pmatrix}^{\iota} = \begin{bmatrix} 0 & a_5 & a_4 & a_3 \\ -a_5 & 0 & a_2 & -a_1 \\ -a_4 & -a_2 & 0 & a_0 \\ -a_3 & a_1 & -a_0 & 0 \end{bmatrix}$$

The six coordinate axes with pointwise and planewise Plücker-coordinates are:

$$\begin{aligned}
X_0 \vee X_1 &= (1:0:0:0:0:0) = \iota[0:0:0:0:0:1] = Y_2 \wedge Y_3 \\
X_0 \vee X_2 &= (0:1:0:0:0:0) = \iota[0:0:0:0:1:0] = Y_1 \wedge Y_3 \\
X_0 \vee X_3 &= (0:0:1:0:0:0) = \iota[0:0:0:1:0:0] = Y_1 \wedge Y_2 \\
X_1 \vee X_2 &= (0:0:0:1:0:0) = \iota[0:0:1:0:0:0] = Y_0 \wedge Y_3 \\
X_1 \vee X_3 &= (0:0:0:0:1:0) = \iota[0:1:0:0:0:0] = Y_0 \wedge Y_2 \\
X_2 \vee X_3 &= (0:0:0:0:0:1) = \iota[1:0:0:0:0:0] = Y_0 \wedge Y_1
\end{aligned}$$

Exercise. Classify the lines in space, i.e. prove the following. If all but 1 coordinate of a line l vanishes, then l is a coordinate axis. If all but 2 coordinates of l vanish, then there is one point X_i and one plane Y_j such that $X_i \prec l \prec Y_j$. If 3 coordinates of l vanish and 3 are $\neq 0$, then either l contains one point X_i or it is in one plane Y_j. If 2 coordinates vanish then l meets two axes but contains no X_i and is in no Y_j. If only 1 coordinate vanishes then l meets only one axis and contains no X_i and is in no Y_j. If no coordinate vanishes l is skew to all axes. $\diamond$

1.1.3 Affine Plücker coordinates

Let in affine space be given the distinct vectors/points $P(p_0, p_1, p_2) = (p_0 : p_1 : p_2 : 1)$ and $Q(q_0, q_1, q_2) = (q_0 : q_1 : q_2 : 1)$ and consider the line PQ. Define $V = P - Q = (v_0, v_1, v_2)$ and $W = P \times Q = (w_0, w_1, w_2)$. Plücker himself defined the coordinates of line PQ as

$$(v_0 : v_1 : v_2 : w_0 : w_1 : w_2)$$

in which

$$\begin{array}{rcl|rcl}
v_0 &=& p_0 - q_0 & w_0 &=& p_1q_2 - p_2q_1 \\
v_1 &=& p_1 - q_1 & w_1 &=& -(p_0q_2 - p_2q_0) \\
v_2 &=& p_2 - q_2 & w_2 &=& p_0q_1 - p_1q_0
\end{array}$$

When we compare these with

$$\begin{aligned}
P \wedge_e Q &= (\ p_0q_1 - p_1q_0 : p_0q_2 - p_2q_0 : p_0 - q_0 : p_1q_2 - p_2q_1 : -(p_1 - q_1) : p_2 - q_2\ ) \\
&= (\ w_2 : -w_1 : v_0 : w_0 : -v_1 : v_2\ )
\end{aligned}$$

we see that the two expressions differ only in the ordering and two signs of the coordinates. Hence the two definitions are equivalent as long as no point is at infinity.

1.2 Mutual position of points, lines and planes

Let again $\mathcal{S}$ denote real 3-dimensional projective space, with a fixed system of reference, see figure 2.

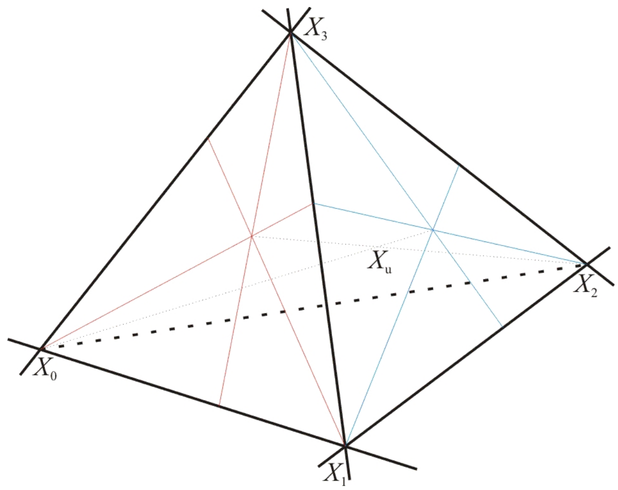

Figure 2: the system of reference

We suppose now that in formulae vectors are denoted vertically (columns) and the transpose operator τ turns horizontal ones into vertical ones and vice versa. But in running text we write vectors horizontally to save space. Point coordinates are in round brackets, dual or plane ones in square brackets. Let $P = (p_i) = (p_0 : p_1 : p_2 : p_3)$, $Q = (q_i) = (q_0 : q_1 : q_2 : q_3)$ and $R = (r_i) = (r_0 : r_1 : r_2 : r_3)$ be points, let l, m, n be lines and let $A = [a_i] = [a_0 : a_1 : a_2 : a_3]$, $B = [b_i] = [b_0 : b_1 : b_2 : b_3]$ and $C = [c_i] = [c_0 : c_1 : c_2 : c_3]$ be planes of $\mathcal{S}$.

Each line has four numerical representations: as a 6-vector or as a 4×4-matrix, pointwise or planewise. E.g.

$$m = (m_k) = (m_{ij}) = [m'_{ij}]^\iota = [m'_k]^\iota$$

For each line the *line conditions* holds:

$$Pl_n(m) \;=\; m_{01}m_{23} + m_{02}m_{31} + m_{03}m_{12} \;=\; m_0m_5 + m_1m_4 + m_2m_3 = 0$$

We list the formulas to compute meet and join in $\mathcal{S}$. We will need the following bilinear function, where l, m are lines

$$\Omega(l, m) = [l_i]^\tau(m_i) = \Sigma_0^5 l_i m_{5-i} = l_0m_5 + l_1m_4 + l_2m_3 + l_3m_2 + l_4m_1 + l_5m_0$$

Observe that $\Omega(l, l) = 2Pl_n(l) = 0$ for all lines.

- **Two points**

$$P = Q \;\Leftrightarrow\; \exists \mu \in \mathbb{R} \setminus 0 : (p_i) = \mu(q_i)$$

If $P \neq Q$ then $m = P \vee Q$ is a line and

$$Pv(m) = (m_i) = (p_i) \wedge_e (q_j)$$

- **One point and one line**

$$P \prec m \;\Leftrightarrow\; [m'_{ij}](p_j) = \mathbf{0}$$

If $P \not\prec m$ then $A = P \vee m$ is a plane with

$$[a_i] = [m'_{ij}](p_j)$$

- **A One point and one plane**

$$P \prec A \;\Leftrightarrow\; [a_i]^\tau (p_i) = 0$$

- **Two lines**
Two lines m, n are skew if and only if $\Omega(m,n) = [m'_i]^\tau(n_i) \neq 0$. If they are not, first check if $m = n$. If this is not the case they meet in a point $P = m \wedge n$ and their join is a plane $A = m \vee n$. To find A and P take any plane B containing m (take for instance one of the non-zero vectors of the matrix $[m_{ij}]$; or, if you do not know if these vectors are 0, take the sum of two or three of them). If $(n_{ij})[B_j] = 0$ this plane contains n as well and hence $B = A$. Else $(n_{ij})[B_j] = P$. Dually: take any point Q on m. If $[n'_{ij}](Q_j) = 0$ this point is on n as well and hence $Q = P$. Else $[n'_{ij}](Q_j) = A$.

  There is no simple symmetrical formula that determines meeting point and joining plane of distinct coplanar lines.

- **One line and one plane**

$$m \prec A \;\Leftrightarrow\; (m_{ij})[a_j] = \mathbf{0}$$

If $m \not\prec A$ then $P = A \wedge m$ is a point and

$$(p_i) = (m_{ij})[a_j]$$

- **Two planes**

$$A = B \;\Leftrightarrow\; \exists \mu \in \mathbb{R} \setminus 0 : \;[a] = \mu[b]$$

If $A \neq B$ then $A \wedge B$ is a line m with

$$[m'_i] = [a] \wedge_e [b]$$

- **Three points or three planes**
The equation of the plane A through P, Q, R is

$$\det(A, P, Q, R) = 0$$

The common point P of A, B, C is obtained from $\det(P, A, B, C) = 0$.

1.3 Exercises

1. Let be given the points $P(2:4:-1:0)$ and $Q(-2:-1:1:1)$, and the planes $A[2:-1:0:3]$ and $B[-1:5:1:2]$.
a. Verify that both P and Q are in A as well as in B.
b. Determine the Plücker-matix of both $P \vee Q$ and $A \wedge B$.
c. Determine the Plücker-vector from each of these matrices and show that the lines are equal.

2. Let be given the system of reference $X_0 \dots X_u, Y_0 \dots Y_u$ and the line $l(1:2:3:4:5:6)$.
a. Determine the points $l \wedge Y_i$ and the planes $l \vee X_i$ for $i \in \{0,1,2,3\}$.
b. Is the point $P(1:0:5:2)$ on l? If not, determine $P \vee l$.
c. Is l in the plane $A(3:-3:1:4)$? If not, determine the meeting point $l \wedge A$.

3. Let be given the lines $l(-9:6:-2:6:5:2)$ and $m(2:-3:1:2:0:-1)$. Are they coincident or skew? If not, determine meeting point and joining plane.

4. Show that line $P \vee Q$ meets the X_1X_3-axis.

Answers. 1b.

$$P \vee Q = \begin{pmatrix} 0 & 6 & 0 & 2 \\ -6 & 0 & 3 & 4 \\ 0 & -3 & 0 & -1 \\ -2 & -4 & 1 & 0 \end{pmatrix}, \quad A \wedge B = \begin{bmatrix} 0 & -3 & -12 & 9 \\ 3 & 0 & 6 & 0 \\ 12 & -6 & 0 & 18 \\ -9 & 0 & -18 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 & 3 \\ 1 & 0 & 2 & 0 \\ 4 & -2 & 0 & 6 \\ -3 & 0 & -6 & 0 \end{bmatrix}$$

1c. $(6:0:2:3:-4:-1) = \iota[-1:-4:3:2:0:6]$
2a. These points and planes can be derived directly from the above matrices.
2b.

$$l \vee P = (l_{ij})^{\iota}(p_j) = \begin{bmatrix} 0 & 6 & 5 & 4 \\ -6 & 0 & 3 & -2 \\ -5 & -3 & 0 & 1 \\ -4 & 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 33 \\ 5 \\ -3 \\ -9 \end{pmatrix}$$

2c.

$$l \wedge A = (l_{ij})[a_j] = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & -5 \\ -2 & -4 & 0 & 6 \\ -3 & 5 & -6 & 0 \end{pmatrix} \begin{bmatrix} 3 \\ -3 \\ 1 \\ 4 \end{bmatrix} = \begin{pmatrix} 11 \\ -19 \\ 30 \\ -30 \end{pmatrix}$$

3. $l \wedge m = (1:-2:2:-1), \quad l \vee m = [1:0:1:3]$

2 The parabolic strip

Definition 2.1 *Given a line l in real projective space, a* parabolic strip *is a projective map*

$$f : \langle \emptyset, l \rangle \rightarrow \langle l, \mathbb{P} \rangle \quad or \quad f : \langle l, \mathbb{P} \rangle \rightarrow \langle \emptyset, l \rangle$$

l is called the axis *or* bearer *of f.*

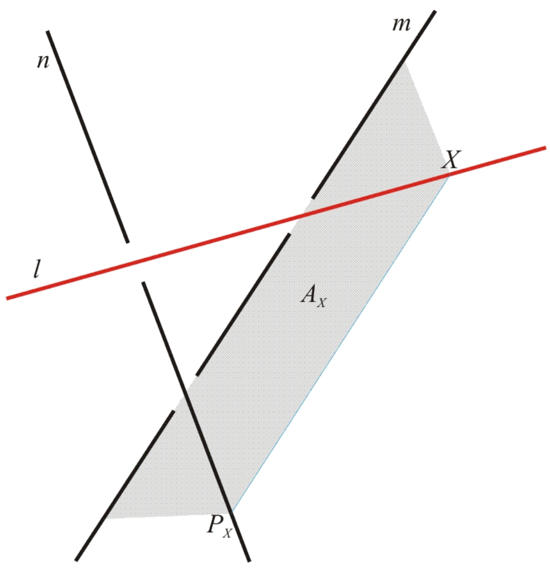

Figure 3: construction of a parabolic strip

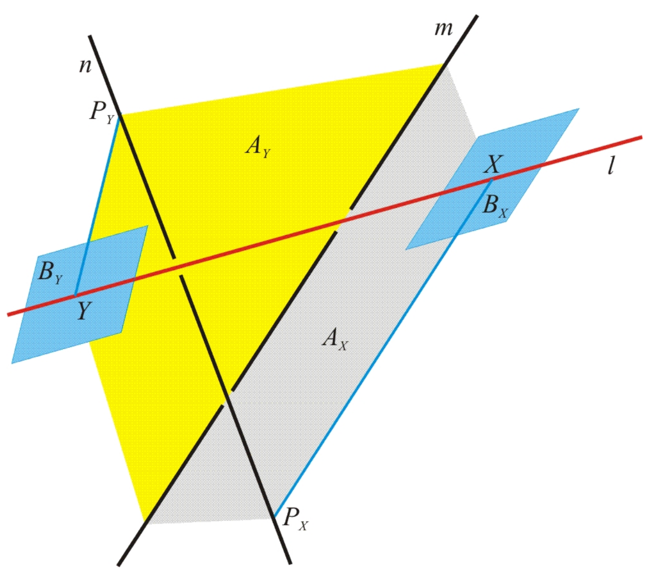

Figure 4: orientation in a parabolic strip

Hence, a parabolic strip maps the points of a line onto the planes through that line, while keeping cross ratios invariant. To put it sloppy: if a point traverses that complete line once, the corresponding plane rotates a half turn around the line.

In figure 3 a standard construction of a parabolic strip is drawn. Start with three pairwise skew lines l, m, n. Take an arbitrary point X on l. The plane $A_X = X \vee m$ meets n in a point P_X. If we define $f(X) = B_X = l \vee P_X = l \vee ((X \vee m) \wedge n)$ then f is projective, hence a parabolic strip. In figure 4 the construction is repeated for the point Y.

Proposition 2.2 Decomposition of parabolic strips. *Given a parabolic strip $f : \langle \emptyset, l \rangle \to \langle l, \mathbb{P} \rangle$ there exist lines m, n, such that $f(X) = ((X \vee m) \wedge n) \vee l$ for each $X \prec l$.*

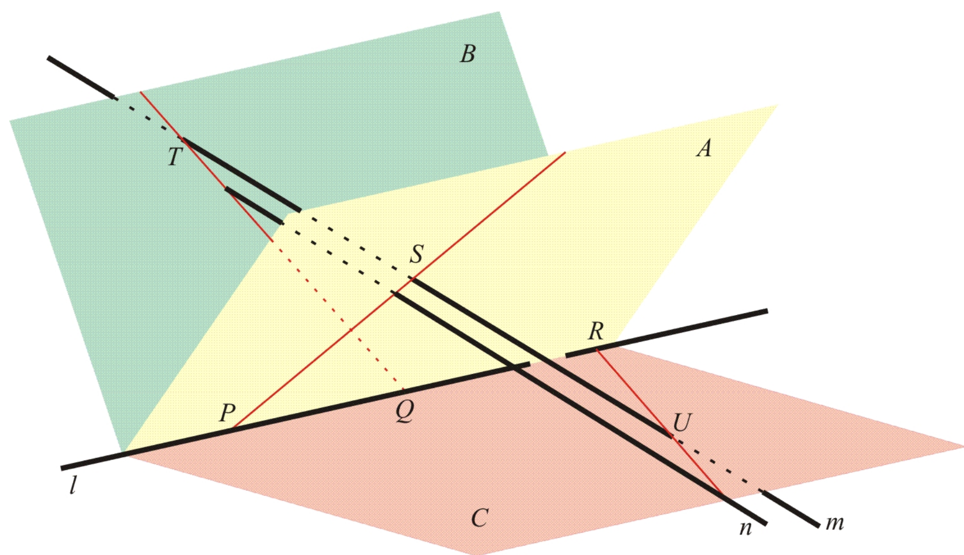

Figure 5: decomposing a parabolic strip

Proof. Take three distinct points P, Q and R on l. Let $A = f(P)$, $B = f(Q)$ and $C = f(R)$, see figure 5. Take an arbitrary line m, skew to l, and let S, T, U be the meeting poinst of m with A, B, C respectively. Observe that the lines PS, QT, RU are pairwiese skew. Take a third point V on line TQ. The plane VPS meets line RU in a point W. Let $n = VW$ and verify that n is skew to both l and m. Define $g(X) = ((X \vee m) \wedge n) \vee l$ for $X \prec l$. Then g and f have equal images on the three distinct points P, Q, R of l, hence they are equal in each point of l. $\diamond$

Since a parabolic strip is a projectivity betwee 1-dimensional spaces, the map is determined by three points and their images. We will denote the strip f by $(PQR\ f(P)f(Q)f(R))$ where P, Q, R are any three distinct points on the axis of f. So the strip of figure 5 is denoted by $(PQR\ ABC)$.

Observe that if in figure 4 you move point Y to the right, the plane $f(Y)$ turns to the left, it is a left-handed orientation[1]. With different positions of m and n this may change. For instance,

[1] An exact definition of 'orientation' involves a good deal of Differential Geometry or Homology Theory, see e.g. [**?**]. In this treatise we will we will handle the subject intuitively.

if you interchange m and n, you will get a right-handed orientation. The rule is as follows. The lines l, m divide the collection of all lines in two parts. Each line of one part can move in that part without crossing either of l, m. If n, n' are in the same part, the corresponding strips have the same orientation, otherwise opposite. See section 3 for an exact treatment of 'orientation'.

3 Orientation

Without proof we state that real projective spaces of odd dimension are orientable, those of even dimension not (see also footnote on page 8). So, our 3-dimensional space is orientable. What does that mean?

Intuitively, moving along a line and simultaneously turning a plane about that line in a clockwise sense is called a right-handed orientation, turning counter clockwise is called left-handed. It is the parabolic strip that does this formally, see figure 6. A parabolic strip is determined by three distinct points on a line and three distinct planes containing that line. So we are looking for a function ω that maps each such 6-tuple $(PQR\ ABC)$ onto

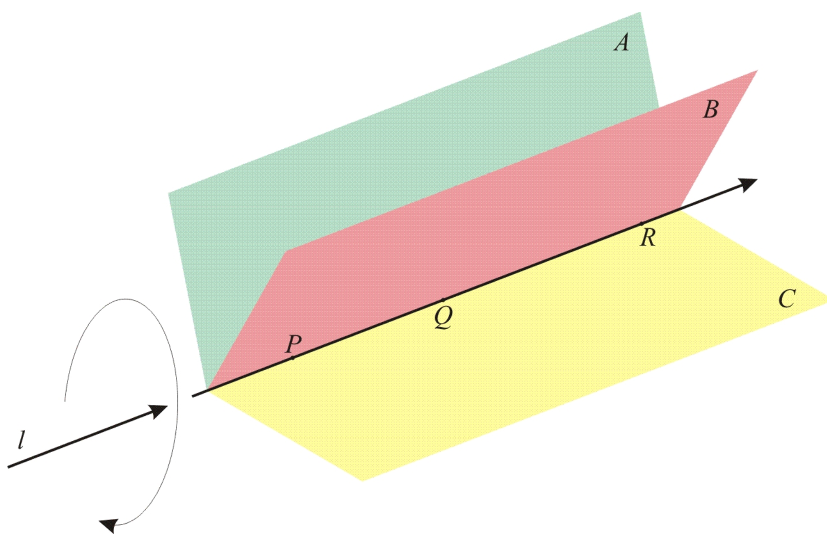

Figure 6: a right-handed parabolic strip

one of the numbers 1 and -1. If p is an even permutation of ABC then $\omega(PQR\ p(ABC))$ should have the same sign as $\omega(PQR\ ABC)$, and if p is odd they must have opposite signs. Similarly $\omega(q(PQR)\ ABC) = \omega(PQR\ ABC)$ if q is an even permutation of PQR, and $\omega(q(PQR)\ ABC) = -\omega(PQR\ ABC)$ if q is odd. In addition, ω should be independent of the choice of the points P, Q, R on l.

We will first fix an initial parabolic strip and assign a positive orientation to it, and then define the orientation of an arbitrary strip relative to the initial one.

Definition 3.1 *The* initial positive *parabolic strip is* $g = (X_0\ X_1\ (X_0+X_1)\ \ Y_2\ Y_3\ (Y_2+Y_3))$. *Its orientation is defined as* $\omega(g) = 1$.

It is shown in figure 7. Observe that it is a right-handed orientation. But if we interchange X_0 and X_1 it becomes left-handed. So we will not use left- and right-handed anymore but only positive and negative with respect to the system of reference, i.e. to the initial positive strip. Observe also that the orientation does not change if we put X_u elsewhere on the line through $X_0 + X_1$ and $X_2 + X_3$ (as long as it does not coincide with one of these points). But if we move it such that $X_0 + X_1$ keeps its position but $X_2 + X_3$ moves to the other segment X_2X_3 then the orientation does change.

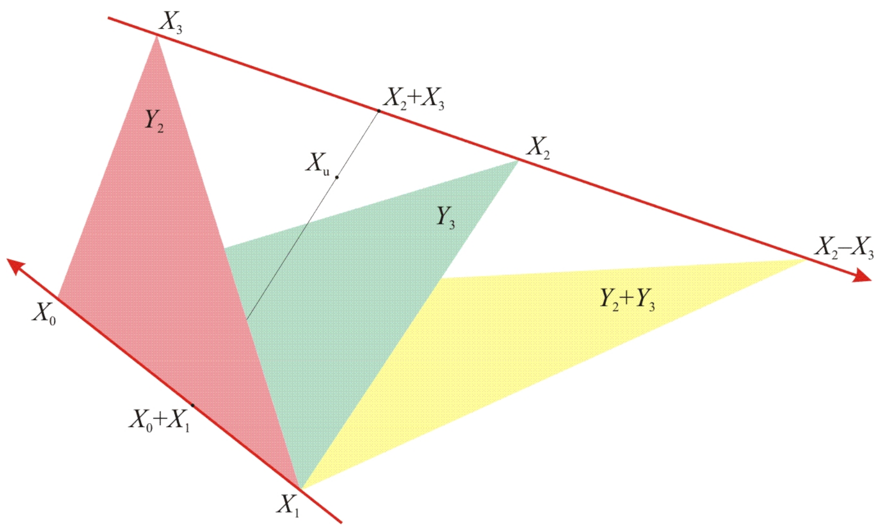

Figure 7: the initial positive orientation

Let $f : \langle \emptyset, l \rangle \to \langle l, \mathbb{P} \rangle$ be any parabolic strip, let P, Q, R be distinct points on line l, and $A = f(P), B = f(Q), C = f(R)$, see figure 8. Let α, β be the (unique) non-zero real numbers

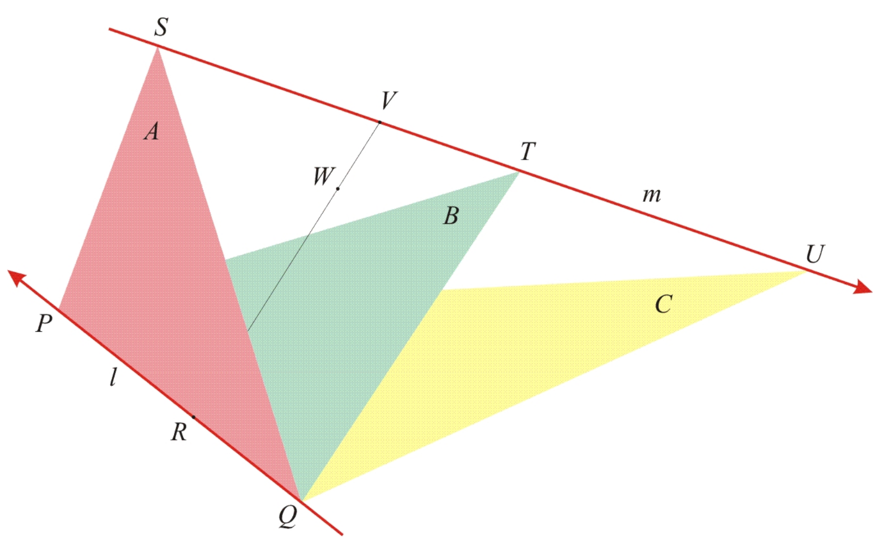

Figure 8: an arbitrary parabolic strip

such that $R = P + \alpha Q$ and $C = A + \beta B$. We will construct a projectivity h that maps $X_0,\ X_1,\ X_0+X_1,\ Y_2,\ Y_3,\ Y_2+Y_3$ on P,Q,R,A,B,C respectively. Let m be any line skew to l and let S,T,U be the meeting points of m with A,B,C respectively. If (M) is the pointwise Plücker-matrix of m then $S=(M)A,\ T=(M)B$ and $U=(M)A+\beta(M)B=S+\beta T$. Define the points $V=(M)A-\beta(M)B=S-\beta T$ and $W=W_\lambda=R+\lambda V$ for any non-zero real number λ. Then P,Q,S,T,W are in general position hence there is a unique projectivity $h=h_\lambda$ that maps X_0,X_1,X_2,X_3,X_u onto P,Q,T,S,W_λ respectively. Let (p_i) be the coordinates of P etc. Then the pointmatrix of h equals

$$\begin{pmatrix} p_0 & \alpha q_0 & -\lambda\beta t_0 & \lambda s_0 \\ p_1 & \alpha q_1 & -\lambda\beta t_1 & \lambda s_1 \\ p_2 & \alpha q_2 & -\lambda\beta t_2 & \lambda s_2 \\ p_3 & \alpha q_3 & -\lambda\beta t_3 & \lambda s_3 \end{pmatrix} = (P, \alpha Q, -\beta\lambda T, \lambda S)$$

Verify that h also maps $X_0+X_1,\ X_2+X_3,\ X_2-X_3$ onto R,V,U respectively. Then also $h(Y_2)=h(X_0\vee X_1\vee X_3)=P\vee Q\vee S=A$, $h(Y_3)=B$ and $h(Y_2+Y_3)=C$. The determinant of h equals $-\alpha\beta\lambda^2\det(PQTS)=\alpha\beta\lambda^2\det(PQST)$.

In section **??** (proposition **??**) we prove that the determinants of $(PQST)$ and $(PQAB)$ have the same sign. So we define:

Definition 3.2 *Let $f:\langle\emptyset,l\rangle\to\langle l,\mathbb{P}\rangle$ be a parabolic strip, let $P,Q,R=P+\alpha Q$ be distinct points on line l, and $A=f(P),B=f(Q),C=f(R)=A+\beta B$. Let m be any line skew to l that meets A,B,C in the points $S=[M]A,\ T=[M]B$ and $U=S+\beta T$ respectively, where $[M]$ is the planewise matrix of m. Then the* orientation *of f is defined as*

$$\omega(f)=\frac{\alpha\beta\det(PQAB)}{|\alpha\beta\det(PQAB)|}=\frac{\alpha\beta\det(PQST)}{|\alpha\beta\det(PQST)|}\ \in\ \{1,-1\}$$

Observe first that a matrix $(PQAB)$ has no meaning in our theory, but its determinant can still be computed. Observe also that interchanging P and Q changes the orientation. And changing the sign of α moves R from one segment of PQ to the other. Similar for interchanging A,B, and for changing the sign of β. That means that ω satisfies the above required conditions of permutation.

Proposition 3.3 *If p is an even permutation of ABC then $\omega(PQR\,p(ABC))=\omega(PQR\,ABC)$. If p is an odd permutation of ABC then $\omega(PQR\,p(ABC))=-\omega(PQR\,ABC)$.* $\diamond$

For completeness we mention that there is of course a dual formula

$$\omega(f)=\frac{\alpha\beta\det(EFAB)}{|\alpha\beta\det(PQAB)|}$$

where $E=P\vee m$ and $F=Q\vee m$. The reader is invited to derive this formula, but we will not use it.

Does these formula comply with $\omega(g)=1$? Obviously $(X_0X_1Y_2Y_3)$ is the identiy matrix and $\alpha=\beta=1$. But $(X_0X_1X_3X_2)$ has determinant -1 and $\beta=-1$: one has to be careful. The problem is that projecitve points P and $-P$ are equal, but in algebraic formulae they may have to be treated differently.

Observe also thet $\omega(f) = \det(h)/|\det(h)|$, i.e. $h = h_\lambda$ (see above) is order preserving if and only if $\det(h) > 0$. As a consequence we have

Proposition 3.4 *Let $f_1 = (P_1Q_1R_1\ A_1B_1C_1)$ and $f_2 = (P_2Q_2R_2\ A_2B_2C_2)$ be parabolic strips and let h be any projectivity that maps $P_1, \ldots C_1$ onto $P_2, \ldots C_2$ respectively. Then $\omega(f_2) = \omega(f_1)\det(h)/|\det(h)|$.* $\diamond$

We still have to prove that the orientation of f does not depend on the choice of P, Q, R.

Proposition 3.5 *Let f be a parabolic strip with axis l, P, Q, R arbitrary distinct points on l, P', Q', R' also arbitrary distinct points on l, A, B, C, A', B', C' their f-images. Then*

$$\frac{\alpha'\beta' \det(P'Q'A'B')}{|\alpha'\beta' \det(P'Q'A'B')|} = \frac{\alpha\beta \det(PQAB)}{|\alpha\beta \det(PQAB)|}$$

where again $R = P + \alpha Q$ and $C = A + \beta B$ and similar for α', β'.

Proof. Take as a local system of reference for points on l $P, Q, P+Q$ and for planes $A, B, A+B$. Then f is linear with matrix

$$f = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Then $f(P) = \alpha A$, $f(Q) = \beta B$ and $f(R) = f(P + \alpha Q) = \alpha A + \alpha\beta B = \alpha(A + \beta B) = \alpha C$. Let the local coordinates be $P' = (p_0 : p_1), Q' = (q_0 : q_1)$, then $R' = (p_0 + \alpha' q_0 : p_1 + \alpha' q_1)$. Now $f(R') = (\alpha p_0 + \alpha\alpha' q_0 : \beta p_1 + \beta\alpha' q_1) = A' + \alpha' B'$, hence – not surprisingly – $\beta' = \alpha'$. The ordering with respect to the primed elements uses

$$\begin{aligned} (\alpha')^2|P'Q'A'B'| = \\ (\alpha')^2|(p_0P + p_1Q)(q_0P + q_1Q)(\alpha p_0A + \beta p_1B)(\alpha q_0A + \beta q_1B)| = \\ (\alpha')^2|(p_0q_1 - p_1q_0)PQ\ \alpha\beta(p_0q_1 - p_1q_0)AB = \\ (\alpha')^2\alpha\beta(p_0q_1 - p_1q_0)^2|PQAB| \end{aligned}$$

which has the same sign as $\alpha\beta|PQAB|$. $\diamond$

Proposition 3.6 *Let l, m be skew lines and let f_l be any parabolic strip with axis l. Then each projective map h that maps l on m defines a parabolic strip $f_m = hf_lh^{-1}$ on m with the same orientation as f_l.*

Proof. Observe first that h maps points on l to points on m and similar for planes; and likewise for h^{-1}. Let X be any point of m. Then $Y = h^{-1}(X)$ is a point of l. Nex $f(Y)$ is a plane containing l and $hf(Y)$ a plane containing m. Since $\det h$ and $\det h^{-1}$ have the same sign $\omega(f_m) = \omega(f_l)$. $\diamond$

Observe the difference with proposition 3.4.

4 Orientation of linear complexes

Regular linear complexes can be considered to consist of a pencil of parabolic linear congruences. Like parabolic strips, regular linear complexes split into two classes of opposite orientation. First we show that all parabolic strips of a regular linear complex have the same orientation.

Proposition 4.1 *If K is a regular linear complex and $m_1, m_2 \in K$. Let f_1, f_2 be the parabolic strips that are the restrictions of n_K to the pointranges on m_1, m_2 respectively. Then $\omega(f_1) = \omega(f_2)$.*

Proof. We have seen that all linear complexes are similar. So we only need to prove our proposition for one complex. Take $K = (k_i) = (0 : 1 : 0 : 0 : 1 : 0) = \iota[0 : 1 : 0 : 0 : 1 : 0]$. So its matrices are

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \iota \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Then $Pl_n(K) = 1$ and $\det n_K = 1 \neq 0$ so K is regular.

Let $m = (m_i)$ be any line of K, then the line condition says $m_0m_5 + m_1m_4 + m_2m_3 = 0$, and $n_K(m) = m$ implies

$$\Omega(k, m) = [k_i]^\tau(m_i) = m_1 + m_4 = 0$$

hence $m_4 = -m_1$. Let the parabolic strip f be the restriction of n_K to the point range on m.

- If $m_3 \neq 0$ then $P(m_1 : m_3 : 0 : -m_5)$, $Q(m_0 : 0 : -m_3 : -m_1)$ and $R = P + Q$ are distinct points on m. Now

$$A = n_K(P) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} m_1 \\ m_3 \\ 0 \\ -m_5 \end{pmatrix} = \begin{bmatrix} 0 \\ -m_5 \\ m_1 \\ -m_3 \end{bmatrix},$$

$$B = n_K(Q) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} m_0 \\ 0 \\ -m_3 \\ -m_1 \end{pmatrix} = \begin{bmatrix} m_3 \\ -m_1 \\ m_0 \\ 0 \end{bmatrix}$$

and $C = n_K(R) = A + B$. Then

$$\det(PQAB) = \begin{vmatrix} m_1 & m_0 & 0 & m_3 \\ m_3 & 0 & m_5 & -m_1 \\ 0 & -m_3 & m_1 & m_0 \\ -m_5 & m_1 & m_3 & 0 \end{vmatrix} =$$

$$= +m_1^2m_3^2 + m_0m_1^2m_5 + m_1^4 + m_0^2m_3^2 + m_3^4 + m_1^2m_3^2 + m_0^2m_5^2 + m_0m_1^2m_5 + m_3^2m_5^2 =$$

$$= +m_1^2m_3^2 + (m_0m_5 + m_1^2)^2 + m_0^2m_3^2 + m_3^4 + m_1^2m_3^2 + m_3^2m_5^2 > 0$$

(since by hypothesis $m_3 \neq 0$), independent of the values of the m_i.

- If $m_3 = 0 \neq m_1 = -m_4$ then take $P(0 : m_0 : m_1 : m_2)$ and $Q(m_0 : 0 : 0 : -m_1)$. Then $|PQAB| = (m_0^2 - m_2^2)^2 + m_1^4 > 0$.

- If $m_1 = m_3 = m_4 = 0$ and one of the remaining $m_i \neq 0$ it is left as an exercise for the reader to show that $|PQAB| > 0$.

This holds for all lines of K, hence all its parabolic strips have the same orientation. $\diamond$

Proposition 4.2 *If K is a linear complex with null-polarity n_K and Plücker-number $p = k_0k_5 + k_1k_4 + k_2k_3$ then $\omega(n_K) = -p/|p|$.*

Let $[K]$ and (K) be the plane- and point-matrix of n_K respectively. At least one of the $k_i \neq 0$, say $k_5 \neq 0$. Define the planes $A = n_K(X_0) = [0 : -k_5 : -k_4 : -k_3]$ and let $P = (0 : -k_4 : k_5 : 0) \neq X_0$ a second point of A. Then the line $m = X_0 \vee P$ belongs tot the pencil $\langle X_0, n_K(X_0)\rangle$ hence to K. Define $B = n_K(P) = [0 : k_2k_5 : k_2k_4 : -k_1k_4 - k_0k_5]$. Now

$$\det(X_0PAB) = \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -k_4 & -k_5 & k_2k_5 \\ 0 & k_5 & -k_4 & k_2k_4 \\ 0 & 0 & -k_3 & -k_1k_4 - k_0k_5 \end{array}\right] = -\left|\begin{array}{ccc} -k_4 & k_5 & k_2k_5 \\ k_5 & k_4 & k_2k_4 \\ 0 & k_3 & -k_1k_4 - k_0k_5 \end{array}\right] =$$

$$k_4\left|\begin{array}{cc} k_4 & k_2k_4 \\ k_3 & -k_1k_4 - k_0k_5 \end{array}\right] + k_5\left|\begin{array}{cc} k_5 & k_2k_5 \\ k_3 & -k_1k_4 - k_0k_5 \end{array}\right] =$$

$$= -k_4^2(+k_1k_4 + k_0k_5 + k_2k_3) - k_5^2(+k_1k_4 + k_0k_5 + k_2k_3) = -p(k_4^2 + k_5^2)$$

By definition 3.2, using $\alpha = \beta = 1$ again, follows the proposition.
If $k_5 = 0$ then there is another $k_i \neq 0$. It is left as an exercise for the reader that again our proposition holds. $\diamond$

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