1 The theorem of Sylvester

Lou de Boer, October 2023

Theorem 1.1 *(J.J. Sylvester, 1814-1897)*
Let be given the distinct planes A and B with common line l.
On l are the distinct points P and Q.
Let $g : \langle P, A\rangle \to \langle Q, B\rangle$ be a projective map with $g(l) = l$.
Let $V_x = T^1(x, g(x))$ for each $x \in \langle P, A\rangle$ and let $V = \bigcup_{x\neq l} V_x$.
- Then there exists one and only one linear complex, K, that contains V.
- This linear complex is regular
- and contains in addition the lines of one parabolic congruence, G, with axis l.
- The restriction of its null-polarity to $\langle P, A\rangle$ equals g.
- No other lines belong to it, i.e. $K = G \cup V$.

Proof. Observe that V_l is a special linear complex, hence $V_l \not\subset K$.
1.) **First we classify the lines in V.**
1a.) Let C be any plane neither containig l nor P nor Q.
Define $a = A \wedge C,\ b = B \wedge C$ and $R = l \wedge C$, see figure 1.

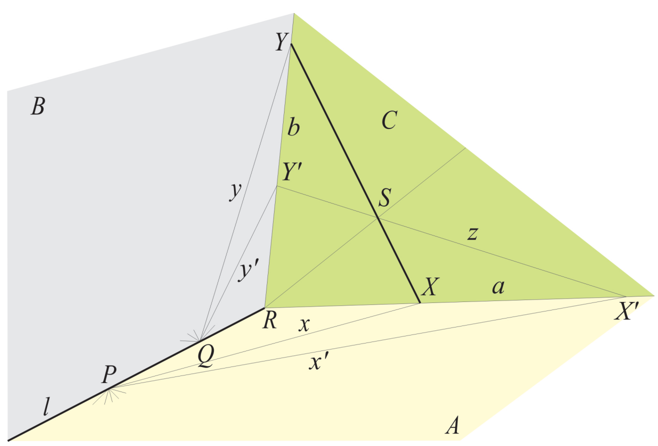

Figure 1: the pencil in an arbitrary plane C

Let x be a line of the pencil $\langle P, A\rangle$.
Let $X = x \wedge a,\ y = g(x)$ and $Y = y \wedge b$.
Define $h : \langle\emptyset, a\rangle \to \langle\emptyset, b\rangle$ by $Y = h(X) = g(X \vee P) \wedge b$ for all $X \prec a$.
Then h is a projectivity, even a perspectivity since $h(R) = R$.
Then there is a point $S \prec C$ such that all lines $X \vee h(X)$ for $X \neq R$ share S, and S is neither on a nor on b.

Let z be any line from pencil $\langle S, C\rangle$ but not SR.
Define $X' = z \wedge a$ and $Y' = z \wedge b$. Then $Y' = h(X') = g(X' \vee P) \wedge b$ and clearly z meets both PX' and $g(PX')$, that is $z \in T^1(PX', g(PX'))$ hence $z \in V$.
So the pencil $\langle S, C\rangle \setminus SR$ belongs to the set V defined by the theorem.

If u is another line of C belonging to V, then there must be an $x \in \langle P, A\rangle$ such that u meets both x and $g(x)$, but then u must pass S.
So, lines in C not belonging to the pencil $\langle S, C\rangle$ do not belong to V.

1b.) If $C = C_P$ does not contain l but does contain P (see left part of figure 2), then $x = C_P \wedge A$ is a line of the first pencil, $\langle P, A\rangle$, and we define $y = g(x)$, $Y = S_P = y \wedge C_P$. Now the entire pencil

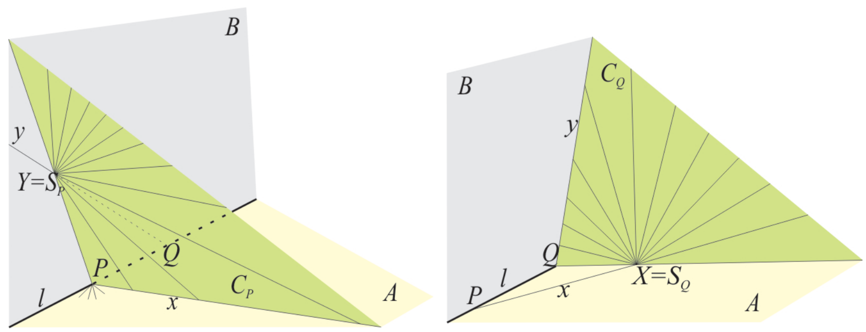

Figure 2: special cases $C \succ P$ and $C \succ Q$

$\langle S_P, C_P\rangle$ belongs to V.
Similar argument if $C = C_Q \succ Q$ but $C_Q \not\succ l$ (use $x = h^{-1}(y)$ in the right part of the figure).
Observe also that if in the left part of figure 2 we rotate plane C_P about x towards A the pencil $\langle S_P, C_P\rangle$ moves towards $\langle Q, A\rangle$, and similarly in the right part of that figure.

1c.) Next trivially the pencils $\langle P, B\rangle$ and $\langle Q, A\rangle$ are subsets of V.
But any other plane through l contains no lines of V exept l itself. For if a line m of such a plane should be in V it must meet a line of each pencil, hence contain P as well as Q, hence being l.

Summary: The following sets are subsets of V: $\langle S, C\rangle \setminus SR$, $\langle S_P, C_P\rangle$, $\langle S_Q, C_Q\rangle$, $\langle P, B\rangle$ and $\langle Q, A\rangle$, for all above defined C, C_P, C_Q.

2.) **The complex**. In figure 3 you see again the double pencil with two lines x, x' from $\langle P, A\rangle$ and their two images $y = g(x)$ and $y' = g(x')$. An extra line through P and in B meets the lines y, y' in the points Y, Y' respectively.
One extra line through Q in A meets x in X, and a second one meets x' in X'.
With two extra lines XY and $X'Y'$ a skew pentagon $(QX, XY, YY', Y'X', X'Q)$ appears, which uniquely determines a regular linear complex K with null-polarity n_K.

3.) K **contains the 'almost pencils'**. Line l belongs to the pencil $\langle Q, A\rangle$ which is part of K because QX and QX' belong to K.
The four independent lines PY, XY, QX and l belong to K, hence the collection of dependent lines of them, viz. the hyperbolic congruence $T^1(x, y)$, is part of K too.
For similar reasons $T^1(x', y') \subset K$.
Let S be any point neither in A nor in B, see figure 4. Let t_1 be the unique transversal from S to x and y. This line belongs to $T^1(x, y)$ hence to K.
The unique transversal t_2 from S to x' and y' belongs to $T^1(x', y')$, hence also to K.
Define $C = t_1 \vee t_2$ and $R = C \wedge l$.
Now the entire pencil $\langle S, C\rangle$ belongs to K, but, exept SR, this pencil belongs to V.
If S lies in A but not on l the two transversals coincide to SQ which is the case of $\langle S_Q, C_Q\rangle$ above

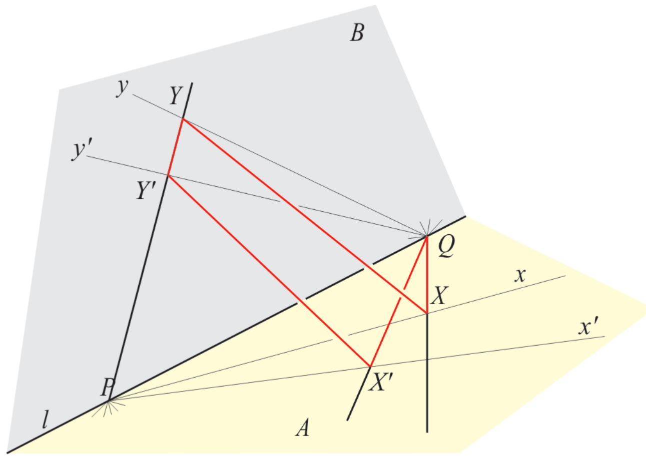

Figure 3: construction of the complex

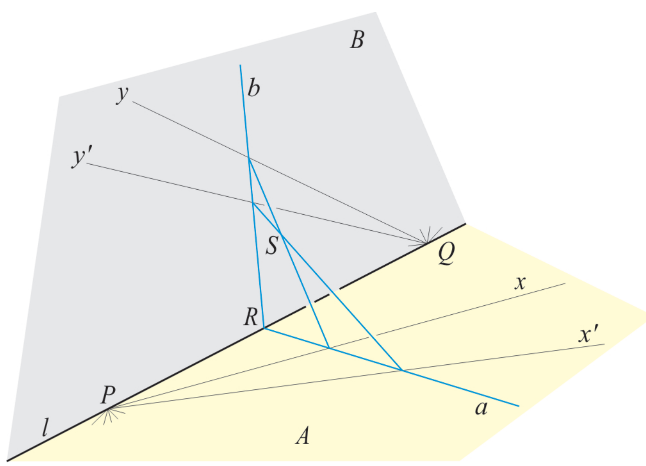

Figure 4: the parabolic congruence

for each $C \succ SQ$.
Similar for $S \prec B$, $S \not\prec l$.
This covers all points S exept those on l, hence all 'almost pencils' of V.

4.) **The congruence**.
Consider the lines PY and QX from figure 3 and line RS from figure 4.
They all belong to K and hence the regulus $T^2(PY, QX, RS)$ is part of K.
Define $D = l \vee S$, then $f_l = (PQR,\ BAD)$ is a parabolic strip.
Since $R = l \wedge C$ we have $n_K(R) = l \vee S = D$, so the restriction of n_K tot the pointrange $\langle \emptyset, l\rangle$ is precisely f_l.
Observe that for each $S \not\prec l$ the entire pencil $\langle R, l \vee S\rangle$ is part of K, so the parabolic congruence G defined by f_l is subset of K.
This congruence also contains all missing lines of the 'almost pencils', hence $V \subset K$.
Since g and the restriction of n_K tot $\langle P, A\rangle$ act the same on l, x, x' they are identical.

5. K **has no other lines**
Suppose $m \in K$. If m meets l in some point R then it must belong to G, otherwise K would contain a bundle of lines with center R. If m is skew to l and meets A in X and B in Y, then – since $n_K(m) = m - g(PX) = n_K(PX) = QY$ and hence $m \in V$. Evidently $K \subset G \cup V$. $\diamond$

See also theorem 21 of [VeblenY1910] from which the essence of this proof is taken.

Exercises a.) Let in the previous configuration D be any plane containing l. Define $R = f_l^{-1}(D)$ and let C be any plane containing R. Investigate how the pencil $\langle S, R \rangle$ changes with C moving about R; also in the cases $D = A$ and $D = B$.
b.) If in the theorem of Sylvester either $P = Q$ or $A = B$ then K is a special complex with axis l. Prove this. What happens when $A = B$ and $P = Q$? $\diamond$

References

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[VeblenY1910] Oswald Veblen and John Wesley Young: *Projective Geometry*, two volumes; Ginn and Company, New York 1910

[Ziegler2012] Renatus Ziegler: *Projective Geometry and Line Geometry*; Dornach 2012